

COMPLEX-SYMMETRIC SPACES

DISSERTATION

**zur Erlangung des Grades
eines Doktors der Naturwissenschaften
der Abteilung für Mathematik
an der Ruhr-Universität Bochum**

von

Ralf Lehmann

aus Berlin

Datum der mündlichen Prüfung: 10.02.88



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Complex-Symmetric Spaces

Ralf Lehmann

Let M be a complete complex Hermitian manifold. Then M is called a Hermitian symmetric space, if and only if for every $x \in M$ there exists a holomorphic isometry of order two which has x as an isolated fixed point.

This condition immediately implies that the group of holomorphic isometries acts transitively on M and that the Hermitian metric is Kählerian.

Moreover, each compact Hermitian symmetric space can be written as a product of a compact torus and a homogeneous projective rational manifold which is Hermitian symmetric. These manifolds are classified by using the classification of semisimple Lie algebras (see e.g. [Hel] and [Ca]).

Generalizations of symmetric spaces to the infinite-dimensional case have turned out to be very interesting (see e.g. [K]).

We are concerned here with manifolds where the isometry condition is dropped. These were first considered by Borel ([Bo]).

Let X be a reduced compact complex space and G a complex subgroup of the complex group of all holomorphic automorphisms. The pair (X, G) is called complex-symmetric with respect to G or Borel symmetric, if and only if for every $x \in X$ there exists $s_x \in G$ of order two such that x is isolated fixed point of s_x . We call s_x a holomorphic symmetry at x .

Our purpose here is to describe the classification of complex-symmetric spaces with an algebraic action of a reductive group.

If (X, G) is complex-symmetric, the group G^0 has an open orbit and the number of orbits is finite. Thus (X, G) is almost G^0 -homogeneous. If we assume in addition that G^0 is acting transitively, then X is Hermitian symmetric ([Bo]).

Of course each compact Hermitian symmetric space is complex-symmetric. Non-homogenous examples can be produced by blowing up appropriate points of Hermitian symmetric spaces.

For example, $\mathbb{P}_1 \times \mathbb{P}_1$ blown up in the four fixed points of the product $(\mathbb{C}^*)^2$ -action is a non-homogeneous complex-symmetric manifold.

Minimal examples can be obtained as follows (cf. part I, § 1).

Let P be a homogeneous $\mathbb{C}^*$ -principal bundle over a compact torus and X the natural compactification of P as a $\mathbb{P}_1$ -bundle. Then X is complex-symmetric, if and only if $P \otimes P$ is holomorphically trivial.

Instead of a $\mathbb{C}^*$ -principal bundle one can take a homogeneous $\mathbb{C}$ -principal bundle. In this case the compactification as a $\mathbb{P}_1$ -bundle is always complex-symmetric.

Note that in these cases the Albanese fibration is not trivial.

Since almost-homogeneous manifolds of dimension two are well known we easily obtain a classification of two-dimensional complex-symmetric manifolds (I, Theorem 1.8).

A minimal two-dimensional complex-symmetric manifold is either a Hermitian symmetric space, a Hirzebruch surface, or one of the $\mathbb{P}_1$ -bundles over a torus introduced above. A Hopf surface is never complex-symmetric.

In general, the compactification of a positive line bundle over a Hermitian symmetric space is complex-symmetric (I, Prop. 1.5).

The examples stated above indicate that in the Kählerian case it is reasonable to start the examination of complex-symmetric manifolds by looking at the fiber of the Albanese map, i.e. by looking at algebraic manifolds with an algebraic group action.

In order to state our results it is necessary to recall the notion of a torus embedding.

A compact torus embedding X is a compactification of an algebraic torus $T \cong (\mathbb{C}^*)^n$ such that the action of T on itself by translations can be extended to an action of T on X . A compact torus embedding is called complex-symmetric, if X is complex-symmetric with respect to a Lie group G with $G^0 = T$.

The main result of this paper is the following

Theorem:

- (1) Let X be a smooth compact algebraic manifold which is complex-symmetric with respect to a Lie group G . Assume that G^0 is reductive and that the action of G^0 is algebraic. Then X is the direct product of a smooth complex-symmetric torus embedding Y and a Hermitian symmetric space which is homogeneous via a semisimple group.
- (2) Moreover, each smooth complex-symmetric torus embedding can be written as a direct product of the following examples.

$$Y_1 := \mathbb{P}_1; Y_2 := \mathbb{P}_1 \times \mathbb{P}_1 \text{ blown up in two points}$$

$$Y_3 := \mathbb{P}_1 \times \mathbb{P}_1 \text{ blown up in four points}$$

$$Y_4 := Y_3 \text{ blown up in four suitable points. } \square$$

Since Hermitian symmetric spaces are completely classified, this Theorem yields a complete classification in the reductive case.

Moreover, we have the following

Corollary:

Let X be a smooth compact algebraic manifold which is complex-symmetric via a reductive group. Assume that X cannot be blown down equivariantly to a manifold. Then X is homogeneous. $\square$

The first to analyse this situation was Ahiezer ([A]). He assumed that X is complex-symmetric with respect to G with G^0 semi-simple. It is shown that there exists a decomposition $G^0 = S_1 \cdot \dots \cdot S_k$ such that the S_i -orbits in X are either Stein or compact Hermitian symmetric spaces.

Using the classification of two orbit varieties he showed that for $k = 1$ and X not G^0 -homogeneous the only possibility is the compact quadric Q_n on which $SO(n, \mathbb{C})$ acts with two orbits ([A]). But Q_n is homogeneous via $SO(n+1, \mathbb{C})$.

In the proof of the Theorem we also first consider the case where X is complex-symmetric with respect to G with G^0 semi-simple, $k > 1$ (I, § 2).

The classification is carried out inductively. Using geometric methods we show that one only has to consider the case where the open G^0 -orbit is Stein. It follows that the closure of each G^0 -orbit is smooth and that the natural biholomorphic mapping from the open G^0 -orbit to a product of affine quadrics can be extended to a biholomorphic map from X to a product of compact quadrics. Such a product is a Hermitian symmetric space.

For the above results the assumption on X and on the G -action can be weakened. We only need that X is smooth or that X is a normal variety with an algebraic G -action.

In § 3 of part I we prove that a complex-symmetric manifold with respect to a reductive group G^0 is the product of a Hermitian symmetric space and a complex-symmetric torus embedding. The methods used here are similar to those used in § 2.

In part II we prove the second part of the above Theorem, i.e. we classify all smooth complex-symmetric torus embeddings.

This classification makes use of the fundamental fact that a torus embedding which is a normal variety can be described by systems of convex rational cones satisfying certain conditions ("fans"). This description is explained in (II, § 1.1 and § 1.2).

A necessary and sufficient condition for a non-singular fan Σ to define a smooth complex-symmetric torus embedding can be stated as follows (II, Theorem 24)

For each cone $\sigma \in \Sigma$ there exists a fan isomorphism φ_σ of order two such that

$$\varphi_\sigma|_\sigma = \text{Id} \quad \text{and}$$

the induced isomorphism on the fan defined by the torus embedding $\overline{\text{orb}(\sigma)}$ is $-\text{Id}$:

By direct calculations this condition implies a classification in dimensions two and three.

For the general classification we show that the system of vectors defining the fan of a complex-symmetric torus embedding is the root system of a Coxeter group, i.e. a finite group generated by reflections. The corresponding variety is the product of lower-dimensional varieties, if and only if the Coxeter group is reducible. Proceeding by induction we prove that an irreducible Coxeter group cannot define a complex-symmetric torus embedding except in dimension two. This proof is based on the classification of Coxeter groups (e.g. [6-B]).

Finally we show that in the singular case new examples arise.

Remark:

It seems very likely that our results can be generalized to normal varieties. However, some of the arguments will probably be more technical.

Acknowledgements:

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Part 1:

C L A S S I F I C A T I O N O F
C O M P L E X - S Y M M E T R I C S P A C E S
W I T H A N A C T I O N
O F A R E D U C T I V E G R O U P

§ 1. Examples of Complex-symmetric Spaces

Of course each Hermitian symmetric space is complex-symmetric. Therefore tori, projective spaces and Grassmannians are complex-symmetric (cf. [Hel], p. 354).

Before giving some non-homogeneous examples we need a basic Lemma (which is implicitly contained in [Bol]).

Lemma 1.1:

Assume (X, G) is complex-symmetric. Consider the normalizer-fibration of the open orbit $\Omega := G^0/H$

$$G^0/H \xrightarrow{N/H} G^0/N, \text{ where } N := \text{Norm}_{G^0}(H^0) .$$

Then N^0/H is an abelian group, the symmetries of Ω respect the normalizer-fibration and induce mappings of the form

$$g \mapsto g_0 g^{-1} \quad (g, g_0 \in N^0/H) \quad \text{on } N^0/H .$$

Proof:

If s_0 is a symmetry for $x_0 \in \Omega$, then the mapping

$$j_0 : G^0 \rightarrow G^0 ,$$

defined by

$$j_0(g) := s_0 g s_0$$

is an automorphism with

$$s_0(g x_0) = j_0(g) x_0 \quad (*) .$$

This automorphism leaves H, H^0, N and N^0 invariant. Therefore j_0 induces an automorphism $\tilde{j}_0$ of the complex group N^0/H^0 . Now $T_e(N^0/H^0)$ can be identified with a subspace of $T_{x_0}(X)$ and (*) yields that the differential of $\tilde{j}_0$ in e is $-\text{Id}$.

By uniqueness of the exponential mapping we have that $\tilde{j}_0$ is the mapping $g \mapsto g^{-1}$; therefore N^0/H^0 and $N^0/H = N^0/H^0/H/H^0$ are abelian.

Now because of (*), for $g \in N^0$ we have:

$$s_0(gx_0) = g^{-1}x_0$$

(This does not imply $j_0(g) = g^{-1} \forall g \in N^0$!), and the Lemma is proved. $\square$

From the proof of 1.1 the following is immediate.

Corollary 1.2:

If Ω is of the form G^0/Γ (Γ discrete), then Ω is an abelian Lie group and the symmetries of Ω are of the form $g \mapsto g_0g^{-1}$ ($g, g_0 \in \Omega$). $\square$

We now want to consider a class of compact manifolds which are almost homogeneous, but not homogeneous, and where the lack of homogeneity has nothing to do with modifications.

Let X be a $\mathbb{P}_1$ -bundle over a complex torus T . Then X is Kähler ([H-0]; p. 92), and T is the Albanese torus of X . It is known that the bundle is holomorphically trivial, iff X is homogeneous.

So we assume that X is almost G^0 -homogeneous, where $G^0 := \text{Aut}_0(X)^0$, but not homogeneous. Then the bundle X is topologically trivial and G^0 is abelian ([H-O], p. 93; [Ma 1]). Either the open orbit Ω is a $\mathbb{C}^*$ -principal bundle over T and G^0 has three orbits (two of them divisors), or Ω is a $\mathbb{C}$ -principal bundle and G^0 has two orbits.

Lemma 1.5:

If Ω is a $\mathbb{C}$ -principal bundle over T then X is complex-symmetric.

Proof:

If $\Gamma < \mathbb{C}^n$ is the lattice defining T , then there exists a homomorphism $\rho : \Gamma \rightarrow \mathbb{C}$ such that

$$\Omega = \mathbb{C}^{n+1}/\tilde{\Gamma} \text{ and } \tilde{\Gamma} := \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma\} \text{ ([Ma 1])}.$$

We extend the equivalence-relation defined by $\tilde{\Gamma}$ to an equivalence-relation $\sim$ on $\mathbb{C}^n \times \mathbb{P}_1$, defined by

$$(z, [x_0 : x_1]) \sim (z', [x'_0 : x'_1]) \Leftrightarrow z' = z + \gamma \wedge [x'_0 : x'_1] = [x_0 + \rho(\gamma)x_1 : x_1] .$$

Then $X = \mathbb{C}^n \times \mathbb{P}_1 / \sim$ and $E := \{[z, [1:0]]\}$ is the exceptional orbit (here $[z, [1:0]]$ denotes the equivalence-class of $(z, [1:0])$).

The main point now is that the symmetries for the open orbit can be extended to automorphisms of X , which are symmetries for E as well. Define

$$\sigma([z, [x_0 : x_1]]) := [-z, [-x_0 : x_1]] .$$

Then σ is well-defined, holomorphic, and an involution. Moreover, the set of fixed points is

$$\{[\gamma/2, [1:0]] | \gamma \in \Gamma\} \cup \{[\gamma/2, [\rho(\gamma)x_1:2x_1]] | x_1 \neq 0, \gamma \in \Gamma\}$$

and therefore discrete. Since we have fixed points in each orbit the Lemma is proved. $\square$

If we compactify a $\mathbb{C}^*$ -principal bundle Ω over T to a $\mathbb{P}_1$ -bundle X by adding the zero-section E and the infinity-section $\tilde{E}$, we obtain two line bundles.

$$L := E \cup \Omega \quad \text{and} \quad L^* := \tilde{E} \cup \Omega$$

which are dual.

Lemma 1.4:

The manifold X is complex-symmetric if and only if L and L^* are equivalent bundles.

Proof:

Assume X is complex-symmetric.

The Albanese fibration $\pi : X \rightarrow T = \text{Alb}(X)$ is $\text{Aut}_0(X)$ -equivariant (see e.g. [H-O], p. 84). Therefore each symmetry s_x of X defines a symmetry for $\pi(x) \in T$. Moreover s_x induces a symmetry on the fiber $\pi^{-1}(\pi(x))$.

So a symmetry for $x \in \Omega$ has to exchange E and $\tilde{E}$ (since all symmetries of $\mathbb{C}^*$ exchange the ends of $\mathbb{C}^*$). Of course a symmetry for $y \in E$, with $\pi(y) = \pi(x)$, stabilizes E and $\tilde{E}$. Since there exists only one symmetry for $\pi(x) \in T$, s_y and s_x induce the same symmetry on T . Therefore the automorphism $s_y \circ s_x$ exchanges E and $\tilde{E}$ and yields a biholomorphic mapping $L \rightarrow L^*$ which is a bundle automorphism. So $L \cong L^*$ is necessary for X to be symmetric.

Now assume $L \cong L^*$ (as bundles). As in Lemma 1.3 we have

$X = \mathbb{C}^n \times \mathbb{P}_1 / \sim$, where

$$(z, [x_0 : x_1]) \sim (z', [x'_0 : x'_1]) \Leftrightarrow z' = z + \gamma \wedge [x'_0 : x'_1] = [x_0 : \alpha(\gamma) \cdot x_1]$$

($\alpha : \Gamma \rightarrow \mathbb{C}^*$ a homomorphism).

We have

$$L \cong L^* \Leftrightarrow \alpha(\gamma) = \alpha(-\gamma) \quad \forall \gamma \in \Gamma.$$

Thus the mapping

$$[z, [x_0 : x_1]] \mapsto [-z, [x_1 : x_0]]$$

is well-defined of order two and yields a symmetry for the open orbit. A symmetry for a point in E (resp. $\tilde{E}$) is defined by

$$[z, [x_0 : x_1]] \mapsto [-z, [-x_0 : x_1]] \quad . \quad \square$$

Remark:

If X is a symmetric $\mathbb{P}_1$ -bundle over a torus T and the open orbit P is a $\mathbb{C}^*$ -principal bundle, then the open orbit has non-constant holomorphic functions and is holomorphically convex. This need not be true if P is a $\mathbb{C}$ -principal bundle. For $\dim_{\mathbb{C}} X \geq 3$ the open orbit can be a Cousin group, i.e. an abelian group without non-constant holomorphic functions (For lower dimensions a Cousin group situation is not possible). For example consider $P := \mathbb{C}^3 / \Gamma_4$, where

$$\Gamma_4 := \langle (1, 0, i), (i, 0, \sqrt{2}), (0, 1, i), (0, i, \sqrt{3}) \rangle_{\mathbb{Z}}.$$

Then P is a Cousin group and a $\mathbb{C}$ -principal bundle over

$$T := \mathbb{C}^2 / \langle e_1, ie_1, e_2, ie_2 \rangle_{\mathbb{Z}} \quad . \quad \square$$

Until now we have obtained our examples by compactifying principal bundles over complex tori. Tori are of course Hermitian symmetric.

From the point of view of curvature another class of Hermitian symmetric spaces are certain homogeneous projective rational manifolds.

So let $Q := S/P$ be a homogeneous projective rational manifold (where $S := \text{Aut}_0(Q)^0$ is semisimple and P parabolic) and P a non-trivial $\mathbb{C}^*$ -principal bundle over Q .

By compactifying P as a $\mathbb{P}_1$ -bundle over Q , we obtain an almost homogeneous manifold X . This manifold contains two line bundles L and L^* over Q , and we choose L such that $\Gamma_0(Q, L^*) = 0$. The zero-section of L is denoted by N , the zero-section of L^* is denoted by E .

The fibration $\pi : X \rightarrow Q$ is automatically $\text{Aut}_0(X)^0$ -equivariant ([H-O], p. 14). Since $H^1(Q, \mathcal{O}^*) \cong H^2(Q, \mathbb{Z})$ is discrete, all elements of S stabilize L and therefore induce automorphisms of X . So the canonically defined mapping

$$\rho : G := \text{Aut}_0(X)^0 \rightarrow S \quad \text{is surjective,}$$

and there exists a semisimple subgroup $\tilde{S} < G$ such that $\rho|_{\tilde{S}}$ is surjective. The group $\tilde{S}$ is maximal semisimple, S and $\tilde{S}$ have isomorphic Lie algebras, and $\tilde{S}$ has three orbits: P, E and N .

Now one has a natural $\mathbb{C}^*$ -action on P which is fiber-preserving and which induces a $\mathbb{C}^*$ -action on X . It can be shown that $\mathbb{C}^*$ is the maximal reductive part in $\ker(\rho)$ (e.g. [H-O], p. 99).

Finally there is an action of $U := \Gamma_0(Q, L)$ on L (for $\sigma \in U$ define $\tilde{\sigma}(x) := x + \sigma(\pi(x))$) which can trivially be extended to X . It turns out that U is the unipotent radical of G and that

$$G \cong (\tilde{S} \cdot \mathbb{C}^*) \ltimes U,$$

where the semidirect product structure is given by the (irreducible) representation of $\tilde{S} \cdot \mathbb{C}^*$ on the vector space of holomorphic sections.

Of course there are two different cases:

Case 1: If $\Gamma_0(Q, L) = 0$, then G is reductive and has three orbits: P, N and E .

Case 2: If $\Gamma_0(Q, L) \neq 0$, then G has only two orbits: the line bundle L and E .

In both cases E is a closed G -orbit which is biholomorphic to Q . So if X is complex-symmetric, there exist symmetries for E (and for Q). By homogeneity Q is Hermitian symmetric.

We are now able to prove

Proposition 1.5:

If Q is symmetric and $\Gamma_0(Q, L) \neq 0$, then X is complex-symmetric.

Proof:

Let $q \in Q$ and σ be a symmetry for q in Q . Then $\sigma \in S$ ([Hel], p. 305) and therefore σ induces a bundle-automorphism

$\tilde{\sigma}$ of X which stabilizes E . Since $\tilde{\sigma}(N)$ is a section of L , by applying an appropriate translation we can assume that $\tilde{\sigma}$ also stabilizes N . Then $p_1 := N \cap \pi^{-1}(q)$ and $p_2 := E \cap \pi^{-1}(q)$ are fixed points of $\tilde{\sigma}$.

Therefore $\tilde{\sigma}$ induces an automorphism of $\pi^{-1}(q) \cong \mathbb{P}_1$ which fixes the points $[1:0]$ and $[0:1]$. All these automorphisms are of the form

$$[z_0:z_1] \mapsto [z_0:\lambda z_1], \lambda \in \mathbb{C}^*.$$

If we combine $\tilde{\sigma}$ with an appropriate element of the reductive part of $\ker(\rho)$, we obtain an automorphism ψ with $\rho(\psi) = \sigma$, which induces $[z_0:z_1] \mapsto [z_0:-z_1]$ on $\pi^{-1}(q) \cong \mathbb{P}_1$. So the differential of ψ in p_1 (resp. p_2) is $-\text{Id}$. The automorphism ψ^2 induces the identity on Q and stabilizes N, E and $\pi^{-1}(q)$ pointwise.

In each fiber $\pi^{-1}(r)$ a mapping of the form

$$[z_0:z_1] \mapsto [z_0:\lambda(r)z_1] \quad (\lambda(r) \in \mathbb{C}^*) \quad \text{is induced by } \psi^2.$$

A calculation in local coordinates shows that $r \mapsto \lambda(r)$ is well-defined and holomorphic, so $\lambda(r) = \lambda(q) = 1$.

Thus ψ^2 is the identity, and we have found a symmetry for p_1 and p_2 ; by homogeneity there exist symmetries for the other points in $G(p_1) \cup G(p_2) = X$. $\square$

We now show that X is never symmetric in the other case.

Proposition 1.6:

If $\Gamma_0(Q, L) = \Gamma_0(Q, L^*) = 0$, then X is not complex-symmetric.

Proof:

We assume that X is symmetric. If $P = G/H$, then the base of the normalizer-fibration is Q ([H-O], p. 81). By Lemma 1.1 a symmetry $\tilde{\sigma}$ for P is a bundle automorphism and therefore

$$\pi(\tilde{\sigma}(x)) = \sigma(\pi(x)) \quad \text{for a suitable } \sigma \in \text{Aut}_Q(Q).$$

Moreover, $\sigma \in S$ ([Hel], p. 305), since we can always find a Hermitian metric such that σ is a holomorphic isometry and isometries that are symmetries are contained in $\text{Aut}_Q(Q)^O$.

Since symmetries of $\mathbb{C}^*$ exchange the ends of $\mathbb{C}^*$, it follows from Lemma 1.1 that $\tilde{\sigma}$ exchanges N and E .

On the other hand $\tilde{\sigma}$ induces an automorphism $\hat{\sigma}$ of X , which stabilizes N and E (cf. Lemma 1.5).

Now we proceed as in the proof of 1.4. The automorphism $\sigma \circ \hat{\sigma}$ yields a bundle-automorphism $L \rightarrow L^*$, so L^2 has to be trivial.

But $H^1(Q, \mathcal{O}^*) \cong H^2(Q, \mathbb{Z})$ and $H^2(Q, \mathbb{Z})$ has no torsion.

Hence L is trivial, contrary to our assumption. $\square$

We now state some

1.7. Corollaries:

- a) If L is a positive line bundle over a Hermitian symmetric manifold Q , then X is complex-symmetric.
- b) The Hirzebruch-surfaces $\Sigma_n (n \geq 1)$ are complex-symmetric.
- c) If $L = K^{-1}$ (the anticanonical bundle over Q), then X is complex-symmetric.

Proof:

- a) A positive line bundle always has a section, since L is homogeneous in the sense of Bott ([Bot]).
- b) The Σ_n are $\mathbb{P}_1$ -bundles over $\mathbb{P}_1$, which occur as compactifications of the positive line bundles H^n over $\mathbb{P}_1$.
- c) By considering the group action one finds n vector fields $Z_1, \dots, Z_n$ which are linearly independent on a dense subset of Q . Then $Z_1 \wedge \dots \wedge Z_n$ is a holomorphic section of K^{-1} . $\square$

We finish this paragraph by giving a complete classification of complex-symmetric surfaces. This is relatively easy, since there exists a detailed classification of almost homogeneous surfaces due to Potters ([P]).

Theorem 1.8:

Let X be a minimal complex-symmetric surface. Then X is one of the following

- (1) the product of two compact homogeneous Riemann surfaces;
- (2) a two-dimensional complex torus;
- (3) the projective plane $\mathbb{P}_2$;
- (4) the Hirzebruch surfaces Σ_n , $n \geq 2$;
- (5) an almost homogeneous $\mathbb{P}_1$ -bundle over a complex torus T , where the open orbit P is a $\mathbb{C}$ -principal bundle over T ;
- (6) an almost homogeneous $\mathbb{P}_1$ -bundle over T , where P is a $\mathbb{C}^*$ -principal bundle over T with P^2 trivial.

Proof:

(1) - (3) are Hermitian symmetric and therefore complex-symmetric.
Examples (4) - (6) were handled above.

If we look at Potter's classification, the only remaining case to be treated is that of Hopf surfaces.

Lemma 1.9:

A Hopf surface is never complex-symmetric.

Proof:

By definition a compact surface X is a Hopf surface, iff its universal covering is $\mathbb{C}^2 \setminus \{(0,0)\}$, i.e.

$$\lambda = \mathbb{C}^2 \setminus \{(0,0)\} / \Gamma, \text{ where } \Gamma \cong \pi_1(X).$$

It can be shown that a Hopf surface is almost homogeneous, iff Γ is abelian ([H-O], p. 104). Since a Hopf surface is not Kähler, it is not Hermitian symmetric. Since homogeneous and complex-symmetric implies Hermitian symmetric ([Bo]) the homogeneous Hopf surfaces are not complex-symmetric.

The non-homogeneous Hopf surfaces are G -equivariant compactifications of abelian groups G . If we consider the induced action of the universal covering $\tilde{G} \cong \mathbb{C}^2$ of G on $\mathbb{C}^2 \setminus \{(0,0)\}$, the open $\tilde{G}$ -orbit $\tilde{\Omega}$ is either

$$\mathbb{C}^* \times \mathbb{C} \text{ or } \mathbb{C}^* \times \mathbb{C}^*$$

Note that in these examples the open orbit of the universal covering group is not simply connected.

The unicity of the universal cover enables us to lift the symmetries of X as automorphisms of $\mathbb{C}^2 \setminus \{(0,0)\}$ which stabilize the open orbit $\tilde{\Omega} \cong \mathbb{C} \times \mathbb{C}^*$.

Since the symmetries for Ω are induced by the mapping $g \mapsto g^{-1}$, the same is true for the lifted automorphisms. Therefore these automorphisms exchange the ends of the $\mathbb{C}^*$ -part of $\tilde{\Omega}$ and don't induce automorphisms of $\mathbb{C}^2 \setminus \{(0,0)\}$. Contradiction. $\square$

Remark:

From Theorem 1.8 one can easily obtain a complete classification of all complex-symmetric surfaces. The only examples which can be blown up and stay almost homogeneous are the Hirzebruch surfaces, $\mathbb{P}_2$ and $\mathbb{P}_1 \times \mathbb{P}_1$.

On the other hand, the examples mentioned by Borel (i.e. $\mathbb{P}_1 \times \mathbb{P}_1$ blown up at four points and $\mathbb{P}_2$ blown up in one point) are complex-symmetric surfaces which are not minimal (cf. part II, Theorem 29 for further examples).

§ 2. Complex-symmetric Spaces With G^0 Semisimple.

In this paragraph we consider the case where (X, G) is a complex-symmetric space and G^0 is semisimple.

We always assume that X is normal and algebraic and that the action of G on X is algebraic. However, the algebraicity is not necessary for most of our considerations.

Some propositions are stated in a more general form than it is necessary here, but they will be used in § 3 where we consider the case of reductive G^0 .

Now let $x \in X$ and $s_x \in G$ be a symmetry for x . Then there exists a maximal compact subgroup in $G^0 \cup s_x(G^0)$ containing the symmetry s_x . This group has at most two connected components. The identity component of this group is denoted by K , the Lie algebra by k . It follows that K is a maximal compact subgroup of G^0 ([Mo]). Hence it is semisimple.

Of course the symmetry defines an involutive automorphism σ of k , and therefore the K -orbit through x carries the structure of a Riemannian symmetric space of the compact type ([Hel]).

Since k is semisimple, we have a decomposition

$$k = \tilde{k}_1 \oplus \dots \oplus \tilde{k}_m$$

in simple ideals.

A factor $\tilde{k}_i$ is either stabilized by the automorphism σ or two isomorphic factors are exchanged by σ . So we have a decomposition

$$k = k_1 \oplus \dots \oplus k_n$$

in minimal σ -stable ideals (Note that this decomposition depends on the G^O -orbit!).

Denoting the restriction of σ to k_i by σ_i and letting u_i (resp. e_i) be the eigenspaces for the eigenvalues $+1$ (resp. -1) of σ_i , we have

$$k_i = u_i \oplus e_i$$

and the pair (k_i, σ_i) is an irreducible, orthogonal symmetric Lie algebra ([Hel], p. 309).

This implies that the subalgebra u_i contains no non-trivial ideals of k_i . This follows from the decomposition, the fact that K is acting effectively and that $\text{ad}_{k_i}(u_i)$ is acting irreducibly on e_i .

If $g_i := k_i^{\mathbb{C}}$ and G_i is the (normal) subgroup of G^O associated to g_i , then the following Lemma holds.

Lemma 2.1:

Consider the G_i -orbit G_i/H_i through x , define $h_i := \text{Lie}(H_i)$. Then either $h_i = u_i^{\mathbb{C}}$ and the G_i -orbit is a Stein manifold or h_i is a maximal parabolic subalgebra containing $u_i^{\mathbb{C}}$ and the G_i -orbit is a Hermitian symmetric space of the compact type.

Proof:

For the sake of simplicity, the index i is left out during the proof. The K -orbit through x is a Riemannian symmetric space, and therefore the isotropy algebra of the K -action is exactly u ([Hel], p. 173). Thus the complex subalgebra h must contain $u^{\mathbb{C}}$.

Assume $h \supsetneq u^{\mathbb{C}}$. We show that any complex subalgebra u^* , satisfying $u^{\mathbb{C}} \subsetneq u^* \subsetneq k^{\mathbb{C}}$ is maximal in $k^{\mathbb{C}}$.

Assume there exist complex subalgebras u^*, u^{**} of $g = k^{\mathbb{C}} = u^{\mathbb{C}} \oplus e^{\mathbb{C}}$ satisfying

$$u^{\mathbb{C}} \subsetneq u^* \subsetneq u^{**} \subsetneq k^{\mathbb{C}}.$$

Define $e^* := u^* \cap e^{\mathbb{C}}$ and $e^{**} := u^{**} \cap e^{\mathbb{C}}$.

The intersections $e^* \cap e$ (resp. $e^{**} \cap e$) are $\text{ad}_k(u)$ -invariant subspaces of e . The irreducibility of this action implies that these subspaces are trivial. Of course $ie \cap e^*$ and $ie \cap e^{**}$ are trivial by the same argument.

So for $0 \neq Z = X + iY \in e^*$, both $X, Y \in e$ have to be different from zero. Applying again the irreducibility we see that the projection

$$e^* \longrightarrow e, \quad Z \longmapsto \text{Re}(Z)$$

is surjective.

Since $e^{**} \subsetneq e^*$ there exist $Z_1 \neq Z_2 \in e^{**}$ with

$$\text{Re}(Z_1) = \text{Re}(Z_2).$$

But then $ie \subset u^{**}$, contrary to $u^{**} \subsetneq k^{\mathbb{C}}$.

Now if $u^{\mathbb{C}} \subsetneq h$, then h is maximal in g . Therefore H is either reductive or parabolic ([Hu], p. 187). Since u is maximal in k and $h \neq u^{\mathbb{C}}$, h is parabolic.

Since the quotient of two reductive groups is a Stein manifold ([Ma 2]) and a homogeneous complex-symmetric space is Hermitian symmetric ([Bol]), the Lemma is proved. $\square$

Remark:

Note that the symmetry s_x induces a symmetry for the G_1 -orbit at x .

Now look at a G^O -orbit G^O/H . Then for $h := \text{Lie}(H)$ we have the inclusion

$$h \supset h_1 \oplus \dots \oplus h_n.$$

Considering the adjoint representation φ_i of $U_i := \text{Iso}_X(k_i)$ on $\text{GL}(e_i^{\mathbb{C}})$, we have the product representation $(\varphi_1, \dots, \varphi_n)$ on $\text{GL}(\bigoplus_{i=1}^n e_i^{\mathbb{C}})$. We claim:

Each subspace of $\bigoplus_{i=1}^n e_i^{\mathbb{C}}$, invariant under the product (*) representation, is of the form $\bigoplus_{i=1}^n \ell_i$, where ℓ_i is a φ_i -invariant subspace.

Since $\ell := h \cap (e_1^{\mathbb{C}} \oplus \dots \oplus e_n^{\mathbb{C}})$ is such a subspace, we have

$$h = \bigoplus_{i=1}^n (u_i^{\mathbb{C}} + \ell_i) \quad \text{and therefore}$$

$$h = \bigoplus_{i=1}^n h_i.$$

So G^O/H^O is of the form $G_1 \dots G_n / H_1 \dots H_n$, where $H_i = G_i \cap H^O$.

Proof of (*):

First we remark that $\dim_{\mathbb{R}} e_i > 1$, otherwise k_i cannot be semi-simple ([Hel], p. 354).

If ℓ_i is a φ_i -invariant subspace of $e_i^{\mathbb{C}}$, consider the induced representation $\tilde{\varphi}_i$ on $\text{GL}(e_i^{\mathbb{C}}/\ell_i)$. Then $\tilde{\varphi}_i$ has no eigenvector except zero.

If $\ell_i = 0$ then the statement is true due to the irreducibility of the $\text{ad}(u_i)$ -action on e_i . So we assume $\ell_i \neq 0$. If there were a non-trivial eigenvector $Z \in \ell_i$, the spaces ℓ_i and

$\ell_i \oplus \langle Z \rangle_{\mathbb{R}}$ would be $\text{ad}(u_i)$ -invariant subspaces satisfying
 $0 \subsetneq \ell_i \subsetneq \ell_i \oplus \langle Z \rangle_{\mathbb{R}} \subsetneq e_i^{\mathbb{C}}$.

Applying the irreducibility as in Lemma 2.1, we obtain

$$\dim_{\mathbb{R}} \ell_i = \frac{1}{2} \dim_{\mathbb{R}} e_i^{\mathbb{C}} \quad \text{and} \quad \ell_i + \langle Z \rangle_{\mathbb{R}} = e_i^{\mathbb{C}}.$$

But then e_i is one-dimensional. So non-trivial eigenvectors cannot exist.

Now consider a subspace ℓ , invariant under the product representation. The following idea is due to Ahiezer. For $Y \in \ell$ define

$$X := \pi_i(Y), \quad \text{where} \quad \pi_i : \ell \rightarrow e_i^{\mathbb{C}}$$

denotes the canonical projection. Then

$$(\varphi_i(g)(X) - X) \in \ell \cap e_i^{\mathbb{C}} \quad \forall g \in U_i$$

$$\Rightarrow \tilde{\varphi}_i([X]) = [X] \quad \text{in} \quad e_i^{\mathbb{C}} / \ell \cap e_i^{\mathbb{C}}$$

$$\Rightarrow X \in \ell \cap e_i^{\mathbb{C}} \quad (\text{there are no non-trivial eigenvectors})$$

$$\Rightarrow \pi_i(\ell) = \ell \cap e_i^{\mathbb{C}}.$$

This completes the proof of (*). $\square$

Consider the decomposition of g in simple ideals

$$g = \tilde{g}_1 \oplus \dots \oplus \tilde{g}_m.$$

If the involutive automorphism σ induced by a symmetry s_x exchanges $\tilde{g}_i$ and $\tilde{g}_j$, then the fixed point set of σ is the diagonal in $\tilde{g}_i \oplus \tilde{g}_j$ and thus

$$\dim_{\mathbb{C}} \tilde{G}_i(x) = \dim_{\mathbb{C}} \tilde{G}_i \tilde{G}_j(x) = \dim_{\mathbb{C}} \tilde{g}_i \quad (x \text{ not necessarily in } \Omega).$$

Therefore the $\tilde{G}_i$ -orbit through x has maximal dimension ($\text{Iso}_x(\tilde{G}_i)$ is discrete), and for each element y of the open orbit Ω the symmetry s_y also exchanges $\tilde{g}_i$ and $\tilde{g}_j$.

So the decomposition $g = g_1 \oplus \dots \oplus g_n$ in minimal ideals which are stable under the symmetries s_x for $x \in \Omega$ is respected by each symmetry of X . Thus

Lemma 2.2:

If (X, G) is complex-symmetric and G^O semisimple, there exists a decomposition $G^O = G_1 \cdot \dots \cdot G_n$ such that for each G^O -orbit G^O/H we have $G^O/H^O = G_1 \cdot \dots \cdot G_n/H_1 \cdot \dots \cdot H_n$ ($H_i := H^O \cap G_i$).

This decomposition is minimal for symmetries of the open orbit. $\square$

In the sequel we will always consider this decomposition.

Lemma 2.3:

Consider a G^O -orbit $G^O/H = G_1 \cdot \dots \cdot G_n/H_1 \cdot \dots \cdot H_n$ and denote the normalizer of H_i^O in G_i by $N_{G_i}(H_i^O)$. Then the factor groups

$$N_{G_i}(H_i^O)/H_i^O \text{ and } N_{G^O}(H^O)/H^O \text{ are finite.}$$

Proof:

It is enough to prove that $N_{G_i}(H_i^O)/H_i^O$ is finite. Since $N_{G_i}(H_i^O)$ is algebraic (e.g. [H-O], p. 31), it is enough to show that $N_{G_i}(H_i^O)/H_i^O$ is discrete.

If the G_i -orbit is compact, then H_i^O is parabolic and thus self-normalizing ([Hu], p. 179). So we can assume that H_i^O is reductive. By Lemma 2.1, $N_{G_i}(H_i^O)$ is parabolic, if $N_{G_i}(H_i^O)/H_i^O$ is positive-dimensional. Moreover H_i^O is its maximal reductive subgroup and

$N_{G_1}(H_1^O)$ is the direct product of its unipotent radical and H_1^O (since H_1^O is normalized). This yields a contradiction. Thus $N_{G_1}(H_1^O)/H_1^O$ is discrete. $\square$

Remarks:

- (1) All considerations above are true for an arbitrary reduced complex space. Neither do we need the normality of X nor the algebraicity of X and of the group action.
- (2) Since the action of G on X is assumed to be algebraic, all isotropies are automatically algebraic. If we drop this assumption, the algebraicity of the isotropies follows directly from Lemma 2.3., since $N_{G_1}(H_1^O)$ is always algebraic ([H-O], p. 31).
- (3) Lemmas 2.1 and 2.3 are known (see [A]).

The next Lemma implies that the classification of complex-symmetric spaces with G^O semisimple can be restricted to the case that the open G^O -orbit Ω is Stein.

Since the Lemma is needed in a more general context later on, we make the following assumptions:

The space X is a complex-symmetric manifold with respect to a group G acting algebraically. Moreover, $G^O = G_1 \cdot \tilde{G}$, where G_1 and $\tilde{G}$ ($\tilde{G}$ any complex Lie group), are normal factors, G_1 semisimple, and all symmetries respect this decomposition.

Then we can show

Lemma 2.4:

Let Ω be the open G^0 -orbit in X , $x \in \Omega$.

If $G_1(x)$ is compact, then X is G^0 -equivariantly biholomorphic to $G_1(x) \times Y$, where Y is a smooth complex-symmetric space with respect to a group $\hat{G}$ with $\hat{G}^0 = \tilde{G}$.

Proof:

Since all symmetries respect the G_1 -orbits in X we can apply the results of Lemma 2.1. Therefore

$$G_1(x) = G_1/P_1, \text{ where } P_1 \text{ is maximal parabolic.}$$

Thus $N_{G_1}(P_1) = P_1$, and we have

$$G_1(x) \cap \tilde{G}(x) = \{x\}.$$

The G_1 -orbits in Ω are all biholomorphic and they are of highest dimension among all G_1 -orbits in X .

Moreover, each maximal compact subgroup K_1 of G_1 acts transitively on each G_1 -orbit in Ω . Therefore the K_1 -orbits in Ω are principal orbits of the K_1 -action on X ([Jän]). The "principal orbit-type theorem" ([Jän]) yields that for each $y \in X$ the isotropy $\text{Iso}_X(K_1)$ is conjugate to a subgroup of $\text{Iso}_Y(K_1)$. However, since $G_1(x)$ is a Hermitian symmetric space (cf. the Remark at the beginning of § 2) the isotropy algebra $\mathfrak{iso}_X(K_1)$ is maximal ([Hel]). Since the compact G_1 -orbits are simply connected a compact G_1 -orbit is either a fixed point of G_1 or equivariantly biholomorphic to a G_1 -orbit in Ω . Linearization of the G_1 -action in a fixed point shows that the G_1 -action on X has no fixed points. Therefore all compact G_1 -orbits are equivariantly biholomorphic.

Since the closure of a non-compact G_1 -orbit contains a compact G_1 -orbit of strictly smaller dimension, non-compact G_1 -orbits do not exist.

Thus $G_1(x) = K_1(x)$ for all $x \in X$, and K_1 only has principal orbits. Thus the equivalence relation on X defined by the G_1 -action is proper. Therefore the orbit space

$$Y := X/G_1 \cong X/K_1$$

carries the structure of a normal complex space induced by the projection

$$\pi : X \rightarrow Y \quad ([Ho], \text{ p. 328}; [Car]).$$

Consider the action of $P_1 = \text{Iso}_X(G_1)$ on X . This action preserves the π -fibers and in each π -fiber there exists only one fixed point of P_1 , since

$$N_{G_1}(P_1) = P_1.$$

The set of fixed points of P_1 is an analytic subset F of X and

$$\pi|_F : F \rightarrow Y = X/G_1$$

is a bijective holomorphic map.

The normality of Y implies that there exists a holomorphic map

$$\tau : Y \rightarrow F$$

which is a holomorphic section.

Thus

$$\Psi : Y \times G_1/P_1 \rightarrow X \quad \text{defined by} \quad \Psi(y, sP_1) := s(\tau(y))$$

is well-defined. By direct calculations one sees that Ψ is biholomorphic and that Y is smooth.

Since all symmetries of X respect the G_1 -part of G^0 the space Y is complex-symmetric with respect to a Lie group $\hat{G}$ with $\hat{G}^0 = \tilde{G}$.

This completes the proof. $\square$

Remarks:

- (1) Assume that we replace "smooth" by "normal" in the above Lemma. Then the Lemma remains valid if we show that all K_1 -orbits are G -equivariantly biholomorphic. This will be done for G^0 semi-simple later on.
- (2) Similar arguments can also be found in ([S]). Again the assumption that G is acting algebraically is not necessary.
- (3) If X is projective algebraic then Y is projective algebraic.

Lemma 2.5:

Let (X, G) be complex-symmetric, X normal, G^0 semisimple. Then there exists exactly one closed G^0 -orbit A .

Proof:

For each non-compact G^0 -orbit E we have $E \cong Q \times \tilde{E}$, where Q is compact and $\tilde{E}$ is Stein. This is due to the fact that if a factor G_1 of G^0 has compact orbits in E , its intersection with orbits of the other factors in E consists of one point only (cf. proof of Lemma 2.4).

Since G^0 is semisimple, $\tilde{E} \cong \mathbb{C}^*$. Thus E has one end in the sense of Freudenthal (e.g. [H-O], p. 55).

Now the Lemma follows from the following general Lemma. $\square$

Lemma :

Let X be a compact complex space, G^0 a complex Lie group acting holomorphically with finitely many orbits. If each non-compact orbit has one end in the sense of Freudenthal then there exists a unique closed orbit in X .

Proof:

We begin by noting that every G^0 -orbit is Zariski-open in its closure. For this, let Ω be a G^0 -orbit and $\bar{\Omega}$ its (analytic) Zariski-closure. Then $\bar{\Omega}$ is irreducible, G^0 -invariant and consists of a finite number of G^0 -orbits. Thus there exists an open orbit F in $\bar{\Omega}$. If $F \neq \Omega$, then Ω lies in the closed analytic subset $\bar{\Omega} \setminus F$, contrary to our assumption.

If we assume that there is more than one closed orbit, then there exists an orbit Y containing more than one closed orbit in its closure and being of minimal dimension among such orbits. Let $A_1, \dots, A_k$ ($k > 1$) be all closed orbits in $\bar{Y} \setminus Y$ and let $B_i \supset A_i$ be the set of all G^0 -orbits in $\bar{Y} \setminus Y$ containing A_i in their closure. Of course B_i is compact, $B_i \cap B_j \neq \emptyset$ ($i \neq j$) and $\bar{Y} \setminus Y = \bigcup_{i=1}^k B_i$ by construction. Therefore Y has $k > 1$ ends. Contradiction. $\square$

Remark:

The same arguments as in the proof of the above Lemma are used in [A] (proof of Theorem 3).

Lemma 2.5 allows us to make the following generalization of Lemma 2.4.

Lemma 2.4':

Let (X, G) be a normal complex-symmetric space with G^0 semisimple. Let $G^0 = G_1 \cdot \dots \cdot G_n$ (Lemma 2.2) and assume that the G_1 -orbits in Ω are compact.

Then $X \cong G_1(x) \times Y$ ($x \in \Omega$),

where Y is a normal complex-symmetric space.

Proof:

By the Remark following Lemma 2.4 it is enough to show that all orbits of a maximal compact subgroup K_1 of G_1 in X are biholomorphic.

If the open orbit Ω is not compact it is holomorphically convex. Thus there exists a G^0 -orbit E of codimension one. Since X is normal, $\Omega \cup E$ is smooth, and we can apply the same arguments as in Lemma 2.4 to show that the K_1 -orbits in E are biholomorphic to the K_1 -orbits in Ω .

If E is compact it is the unique closed G^0 -orbit. Since Ω has one end, it follows that all K_1 -orbits in X are equivariantly biholomorphic.

If E is not compact consider a G^0 -orbit A of codimension two in X which is contained in $\bar{E}$. Since the normalization $v: \tilde{E} \rightarrow \bar{E}$ is a finite map, we can consider $\tilde{E}$, since the K_1 -orbits are simply connected. Here the same arguments as above can be applied in order to show that the K_1 -orbits in A and in Ω are biholomorphic. By repeating this procedure we finally arrive at the unique closed G^0 -orbit. $\square$

From now on we can restrict ourselves to the case where the open G^0 -orbit Ω is Stein.

The following Lemma is very useful in the proof of Lemma 2.8 and in § 3.

Lemma 2.6:

Let H be a positive-dimensional complex Lie group and assume that there is a faithful representation

$$\rho : H \rightarrow \mathbb{C}^* .$$

Then $H = H^0$, i.e. H is connected.

Proof:

The representation $\rho : H^0 \rightarrow \mathbb{C}^*$ is faithful and $\rho(H^0) = \mathbb{C}^*$.

Thus $H = H^0$. $\square$

Now let $G^0 = G_1 \cdots G_n$ (see Lemma 2.2) and let the open orbit $\Omega := G^0/H$ be Stein. Then H is reductive and there exists a G^0 -orbit E of codimension one in X .

Lemma 2.7:

There exists exactly one factor G_i such that

$$\overline{G_i(x)} \setminus G_i(x) \subset E \quad \forall x \in \Omega .$$

Moreover, $\overline{G_i(x)}$ consists of two G_i -orbits and is smooth.

The proof follows immediately from the Lemma 2.8, where we make the following more general assumptions:

The normal space (X, G) is complex-symmetric with respect to G (algebraic action). The group G^0 is the product $G^0 = \tilde{G} \cdot T$ of normal factors, where $\tilde{G}$ is semisimple and T is an algebraic torus (i.e. $T \cong (\mathbb{C}^*)^k$) acting effectively on X .

Since the symmetries respect the decomposition $\tilde{G} \cdot T$, we have a decomposition $\tilde{G} = G_1 \cdot \dots \cdot G_n$ as in Lemma 2.2.

Lemma 2.8:

Let the open G^0 -orbit Ω be Stein and let E be a G^0 -orbit of codimension one in X where the torus T has maximal-dimensional orbits (i.e. the T -orbits in E are isomorphic to T).

Then there exists exactly one factor G_i such that

$$\overline{G_i(x)} \setminus G_i(x) \subset E \quad \forall x \in \Omega.$$

Moreover, $\overline{G_i(x)}$ consists of two G_i -orbits and is smooth.

Proof:

Let $y \in E$ and $J = J_1 \cdot \dots \cdot J_n := (\text{Iso}_y(G^0))^0$.

First we show that J is not reductive.

Assume that J is reductive. Since X is normal, $\Omega \cup E$ is smooth. If we look at the induced representation of J on the tangent space $T_y(X)$, we have

$$T_y(X) = T_y(E) \oplus V,$$

where V is a subspace stable under J with $\dim_{\mathbb{C}} V = 1$ by construction.

There exists a neighbourhood $\tilde{U}$ of the origin in $T_y(X)$, a neighbourhood $U(y)$ of y in X and a biholomorphic mapping $\varphi : \tilde{U} \rightarrow U(y)$ such that

$$U(y) \cap \Omega = \varphi(\tilde{U} \cap \{v \neq 0\}), \quad \text{where } \langle v \rangle_{\mathbb{C}} = V.$$

Since the action of J can be faithfully linearized if $U(y)$ is small enough, there exists a complex subspace L of dimension one in $U(y) \cap \Omega$ which is stable under a maximal compact subgroup of J .

Since $\dim_{\mathbb{C}} J = \dim_{\mathbb{C}} (\text{Iso}_x(G^0)) + 1$ ($x \in \Omega$ arbitrary), the action of J on L cannot be trivial, and for every $x \in L$ it follows that J contains $\text{Iso}_x(G^0)^0$. But then J contains a parabolic factor by Lemma 2.1. Contradiction.

Next we show that exactly one factor of J is parabolic. If we assume that more than one factor is parabolic, there exists a factor G_j which has closed and non-closed orbits of the same dimension, because $\dim J = \dim H + 1$. Of course the G_j -orbits in Ω are of highest dimension.

But each maximal compact subgroup K_j of G_j acts transitively on the compact G_j -orbits ([Mon]). Since the generic K_j -orbits in Ω are of highest dimension they have the same dimension as the G_j -orbits in Ω . Thus the G_j -orbits in Ω are compact. This is absurd.

So we have $J = J_1 \cdot \tilde{J} = J_1 \cdot \tilde{J}_2 \cdots \tilde{J}_n$ with J_1 parabolic and $\tilde{J}$ reductive. Correspondingly $G^0 = G_1 \cdot S \cdot T$.

Now for a maximal torus D_1 in J_1 we find a one-dimensional complex subspace V of $T_y(X)$ such that

$$T_y(X) = T_y(E) \oplus V \quad \text{and}$$

such that $\tilde{J} \cdot D_1$ stabilizes V .

Moreover, by faithful linearization of the $\tilde{J}$ -action it follows that $\tilde{J}$ acts trivially on V .

We now prove that the action of D_1 on V is not trivial.

Proceeding by contradiction, assume that the action of D_1 on V is trivial.

By faithful linearization we find a one-dimensional subset L in Ω near y such that $\text{Iso}_x(G^0) \supset D_1 \cdot \tilde{J}$ for all $x \in L$.

Now let $x, x' \in L$, $x = g(x')$ ($g \in G^0$) and

$$g = t \cdot s = t \cdot s_1 \cdot \dots \cdot s_n, \text{ where } s_i \in G_i \text{ and } t \in T.$$

Since $D_1 \tilde{\mathcal{J}} \subset \text{Iso}_X(G) = g^{-1} \text{Iso}_X(G^0)g$, we have

$$g D_1 \tilde{\mathcal{J}} g^{-1} = (s_1 D_1 s_1^{-1}) (s_2 \tilde{\mathcal{J}}_2 s_2^{-1}) \cdot \dots \cdot (s_n \tilde{\mathcal{J}}_n s_n^{-1}) \in \text{Iso}_X(G^0).$$

On the other hand

$$D_1 \tilde{\mathcal{J}} = D_1 \tilde{\mathcal{J}}_2 \cdot \dots \cdot \tilde{\mathcal{J}}_n \in \text{Iso}_X(G^0),$$

thus

$$s_i \in N_{G_i}(\tilde{\mathcal{J}}_i), \quad i = 2, \dots, n.$$

Moreover, $s_1 D_1 s_1^{-1}$ is a maximal torus in $\text{Iso}_X(G_1)$ and therefore conjugate to D_1 by an element $\tilde{s}_1 \in \text{Iso}_X(G_1)$.

It follows that

$$s_1 = h \tilde{s}_1^{-1} \text{ where } h \in N_{G_1}(D_1).$$

Since the groups

$$N_{G_i}(\tilde{\mathcal{J}}_i)/\tilde{\mathcal{J}}_i, \quad i = 2, \dots, n \quad (\text{resp. } N_{G_1}(D_1)/D_1)$$

are finite by Lemma 2.3 (resp. [Hul]), the set

$$\{s(x) \mid s_i \in N_{G_i}(\tilde{\mathcal{J}}_i), \quad i = 2, \dots, n; s_1 = h \tilde{s}_1^{-1}\}$$

is finite.

Thus it follows that the intersection of L with a suitable T -orbit in Ω is one-dimensional

It follows that γ is contained in the closure of this T -orbit.

Since the T -orbits in Ω and E have the same dimension, we obtain a contradiction to the algebraicity of the T -action on X .

So we have proved that the action of D_1 on V is not trivial.

We claim that there exists exactly one G_1 -orbit

$G_1(x)$ in Ω such that $y \in \overline{G_1(x)}$.

Since D_1 acts as $\mathbb{C}^*$ on L , it follows that there exists $x \in L$ such that $y \in \overline{G_1(x)}$. Since G_1 is the only group with smaller orbits in E than in Ω , the algebraicity of the action shows that G_1 is the unique factor with this property.

Assume that y is contained in the closure of several G_1 -orbits in Ω . Since we have shown that $\text{Iso}_y(S) < \text{Iso}_x(S)$ if $y \in E \cap \overline{G_1(x)}$ this is only possible when the torus T has a finite isotropy in y which is not trivial. By faithful linearization we then find a faithful representation of a complex Liegroup on V . Thus this group is connected by Lemma 2.6, and we have

$$\text{Iso}_y(T) \subset \mathcal{C}.$$

Thus $G_1(x)$ is unique.

We will prove that $\overline{G_1(x)}$ is smooth.

For this, note that D_1 is a maximal torus in J_1 and thus has a finite number of fixed points in $G_1(y)$ ([Hu]). If the set of fixed points of D_1 in $\overline{G_1(x)}$ is positive-dimensional, each positive-dimensional connected component of this fixed-point-set will contain points of E in its closure. This is impossible by our arguments above (D_1 does not act trivially on V !).

Thus D_1 has only a finite number of fixed points in $\overline{G_1(x)}$. Consequently, for x' generic in $G_1(x)$ the set $\overline{D_1(x')}$ will contain each fixed point of D_1 in $\overline{G_1(y)}$, since $G_1(x)$ is Stein. Consider the local coordinates $z_1, \dots, z_N$ ($N := \dim_{\mathbb{C}} X$) in $U(y)$ obtained by faithful linearization. We may assume that

$$U(y) \cap E = \{z_1 = 0\} \cap U(y),$$

$$U(y) \cap ST(y) = \{z_1 = \dots = z_k = 0\} \cap U(y),$$

$$U(y) \cap G_1(y) = \{z_1 = 0, z_{k+1} = \dots = z_N = 0\} \cap U(y),$$

and that the D_1 -action is locally given by

$$\lambda(z_1, \dots, z_N) = (\chi_1(\lambda)z_1, \dots, \chi_k(\lambda)z_k, z_{k+1}, \dots, z_N)$$

$(\chi_i : D_1 \rightarrow \mathbb{C}^*$ any non-trivial character).

Now let $x' \in U(z) \cap G_1(x)$ be a generic point such that $\overline{D_1(x)}$ contains y . Then there is a sequence $(\lambda_\nu)_{\nu \in \mathbb{N}}$ in D_1 such that

$$\lambda_\nu(x') \rightarrow y.$$

If x' has coordinates $(w_1, \dots, w_N)$ we have

$$\lambda_\nu(x') = (\chi_1(\lambda_\nu)w_1, \dots, \chi_k(\lambda_\nu)w_k, w_{k+1}, \dots, w_N) \rightarrow (0, \dots, 0)$$

only if $w_{k+1} = \dots = w_N = 0$.

Thus $G_1(x) \cap U(y)$ is contained in $U(y) \cap \{w_{k+1} = \dots = w_N = 0\}$.

Since both sets have the same dimension

$$\overline{G_1(x)} \cap U(y) = \{z_{k+1} = \dots = z_N = 0\} \cap U(y).$$

Thus $\overline{G_1(x)}$ is smooth.

Moreover, $\overline{G_1(x)}$ is complex-symmetric since all symmetries respect the decomposition

$$G^0 = T \cdot G_1 \cdot \dots \cdot G_n \text{ of } G^0. \quad \square$$

Remark:

In the semisimple case (i.e. T is trivial) the algebraicity of the G -action is not necessary.

Now we consider the compact quadric

$$Q_n := \{[z_0 : \dots : z_n] \in \mathbb{P}_n(\mathbb{C}) \mid z_0^2 = \sum_{i=1}^n z_i^2\} \subset \mathbb{P}_n(\mathbb{C}).$$

This quadric is a Hermitian symmetric space for the group $SO(n+1, \mathbb{C})$, and there is an action of the group $SO(n, \mathbb{C})$ on Q_n (e.g. defined by $A([z_0 : z_1 : \dots : z_n]) = A([z_0 : z]) := [z_0 : Az]$) having two orbits.

The closed orbit is a compact quadric Q_{n-1} (e.g. defined by

$Q_{n-1} := Q_n \cap \{z_0 = 0\}$), and the complement is an affine quadric $Q_{(n)}$.

For every $z \in Q_n$ there is a symmetry respecting this orbit decomposition (consider diagonal matrices acting on $\mathbb{P}_n$). Thus Q_n is complex-symmetric with respect to $G^0 = SO(n, \mathbb{C})$, but not homogeneous.

The following Lemma was proved by Ahiezer for G^0 simple ([A]).

But by Lemma 2.7 it is not difficult to apply the same methods inductively for the case of semisimple G^0 and normal X .

Lemma 2.9:

If $G^0 = G_1 \cdot \dots \cdot G_n$ and if Ω is Stein, then $g_1 \cong so(m_1, \mathbb{C})$, the closed G_1 -orbits are irreducible compact quadrics Q_{m_1-1} and the non-closed G_1 -orbits are affine quadrics $Q_{(m_1)}$.

Proof:

For $n = 1$, X is a normal space with two G_1 -orbits (even if G_1 is not simple (Lemma (2.2)), there cannot be more than two G_1 -orbits), one of them being a divisor.

Thus X is a smooth two-orbit variety having algebraic isotropies. These varieties are classified ([A1]; the classification only needs the algebraicity of the isotropies; in the Kählerian case cf. [H-Sn]). Symmetric varieties only occur for $g_1 = so(m_1, \mathbb{C})$ and we see that $X \cong Q_{m_1}$ with the orbit-structure introduced above ([A]).

Now assume that $G^0 = G_1 \cdot \dots \cdot G_n$. Then by Lemma 2.7 there exists a factor G_1 such that for $x \in \Omega$, $\overline{G_1(x)}$ is a smooth two-orbit variety. Thus $G_1(x)$ is an affine quadric.

Since the closed G_1 -orbits are all simply connected and the closed G^0 -orbit is unique, all closed G_1 -orbits have to be G_1 -equivariantly biholomorphic (cf. Lemma 2.4').

Let E be a G^0 -orbit of codimension one where G_1 has compact orbits. Consider the normalization

$$\nu: \tilde{E} \rightarrow E \text{ of } \tilde{E}.$$

Note that $\tilde{E}$ is perhaps not complex-symmetric, but symmetries for the open orbit in $\tilde{E}$ can be lifted to symmetries for the open orbit in E , since the restriction of ν to the open orbit is biholomorphic. Since ν is finite and G -equivariant, Lemmas 2.1 and 2.2 apply to the connected components of the isotropies of the G -action on $\tilde{E}$. Thus Lemmas 2.8 and 2.4' can be applied to $\tilde{E}$.

It follows that $\tilde{E} \cong Q_{m_1-1} \times \tilde{F}$ and that there exists a factor G_2 in G and $\tilde{x}$ in the open orbit of $\tilde{E}$ such that $\overline{G_2(\tilde{x})}$ is a smooth two orbit variety. Since $G_2(\tilde{x})$ has one end and $\nu|_{G_2(\tilde{x})}$ is biholomorphic we find symmetries for $\overline{G_2(\tilde{x})}$ by lifting suitable symmetries of $\tilde{E}$ to the normalization.

Thus $G_2(\tilde{x})$ is an affine quadric and the compact G_2 -orbits in $\tilde{E}$ and hence in X are compact quadrics.

Since affine quadrics are simply connected, the G_2 -orbits in Ω are also affine quadrics.

Now the same argument can be applied to the other factors. This completes the proof of Lemma 2.9. $\square$

We now want to analyse the orbit-structure of a complex-symmetric space (X, G) with G^0 semisimple.

Lemma 2.10:

Assume that the open orbit

$$\Omega = G_1 \cdots G_n / H_1 \cdots H_n \text{ of } X \text{ is Stein and that } X \text{ is normal.}$$

Then for each $i \in \{1, \dots, n\}$ there exists exactly one G^0 -orbit E_i of codimension one such that the G_i -orbits in E_i are compact.

Proof:

If E_i exists, by Lemma 2.8 $\overline{G_i(x)}$ is smooth and $\overline{G_i(x)} \setminus G_i(x) \subset E_i$ for all $x \in \Omega$ since $G_i(x)$ has one end. Thus E_i is unique.

So we only have to prove the existence of E_i .

Proceeding by contradiction assume that G_i has non-compact orbits in each G^0 -orbit of codimension one. Let $E_1, \dots, E_s$ be the G^0 -orbits of codimension one. By Lemma 2.9 the G_i -orbits in Ω are affine quadrics G_i/H_i .

For $x \in \Omega$ the isotropy subalgebra $\text{iso}_x(G_i)$ is an element of a certain Grasmannian Z .

Consider the mapping

$$\tau : \Omega \rightarrow Z, \quad x \mapsto \text{iso}_x(G_i).$$

This mapping is G_i -equivariant and $\tau(\Omega)$ is an algebraic G_i -orbit in Z which is isomorphic to $G_i/N_{G_i}(H_i)$. Lemma 2.3 implies that $\tau(\Omega)$ is Stein.

Now let $(x_k)_{k \in \mathbb{N}}$ be a sequence in Ω with

$$\lim_{k \rightarrow \infty} x_k = y \in E_j.$$

Consider the sequence $(\tau(x_k))_{k \in \mathbb{N}}$ and let z be a cluster point of this sequence (Z is compact). We have

$$z \subset \text{iso}_y(G_i),$$

but both sets have the same dimension. Thus

$$z = \text{iso}_y(G_i),$$

and the mapping τ can be extended to a holomorphic mapping

$$\tau : \Omega \cup \left(\bigcup_{j=1}^S E_j \right) \rightarrow Z$$

with

$$\tau(\Omega \cup \left(\bigcup_{j=1}^S E_j \right)) = \tau(\Omega) \quad .$$

Since $\tau(\Omega)$ is Stein there exists a non-constant holomorphic function f on $\tau(\Omega)$. Then $f \circ \tau$ is a non-constant holomorphic function on $X \setminus A$ where A is an analytic set of codimension two.

This is absurd since X is compact and normal.

□

We now show that if X is smooth (resp. normal), then the closure of each G^0 -orbit in X is also smooth (resp. normal).

In the non-singular case this can be shown without the assumption that X is algebraic. In the singular case the algebraicity of X and of the G -action on X is essential for our proof.

Lemma 2.11:

If X is smooth then the closure of each G^0 -orbit is smooth.

Proof:

By Lemma 2.4 it is enough to consider the case where the open G^0 -orbit Ω is Stein.

At first we show that the closure of a G^0 -orbit of codimension one in X is smooth.

If we have a smooth action of a compact group on a complex manifold X , then the set of all points where the isotropies are conjugates of a fixed isotropy is smooth ([Jän], p. 5).

If E is a G^0 -orbit of codimension one and if G_1 has compact orbits in E , then G_1 has compact orbits in $\bar{E}$ and the orbits of a maximal compact subgroup K_1 of G_1 in E coincide with the G_1 -orbits in $\bar{E}$ (apply Lemma 2.4' to the normalization $\hat{E}$ of $\bar{E}$ and note that all G_1 -orbits in $\hat{E}$ and hence in $\bar{E}$ are simply connected).

For $y \in E$ let $L_1 := \text{Iso}_y(K_1)$.

We have shown that $\bar{E}$ is contained in the smooth set $\tilde{E}$ of all points $y' \in X$ with $\text{Iso}_{y'}(K_1)$ conjugate to L_1 . If we consider the action of $\text{SO}(n, \mathbb{C})$ on the affine quadric $Q_{(n)}$, the generic orbits of each maximal compact subgroup of $\text{SO}(n, \mathbb{C})$ are real hypersurfaces. Thus each connected component of $\tilde{E}$ has real codimension at least two ([Jän], p. 10). Since $\bar{E}$ is compact and irreducible, it is a connected component of $\tilde{E}$ and therefore smooth.

Now the Lemma can be proved inductively for the lower-dimensional G^0 -orbits since each G^0 -orbit is contained in the closure of a G^0 -orbit of codimension one. $\square$

Before treating the singular case, we need the following

definition (e.g. [B-L-V]); (L-V)):

Let a reductive group G act algebraically on an algebraic variety X . Assume that G has a dense open orbit $\mathcal{O}$. Then X is called spherical if a Borel subgroup B (and hence every Borel subgroup) of G has a dense orbit in X .

It is known that each Borel subgroup of $SO(n, \mathbb{C})$ has a dense orbit in the affine quadric $Q(n)$. Hence Lemma 2.9 implies that a complex-symmetric space (X, G) with G semisimple is spherical.

More generally, it can be proved that the quotient of a reductive group G by the set of fixed points of an involutive automorphism is spherical ([V]).

A summary of some of the basic results on spherical varieties can be found in [B-P].

For example each irreducible G -stable subvariety of a spherical variety is spherical and the groups G and B have only finitely many orbits in X ([Br]).

The following property of spherical varieties is very important for inductive arguments.

The proof can be found in ([B-P], p. 272; Proposition 3.5). We underline that the algebraicity of X and of the G -action on X are of central importance for the proof.

Proposition 2.12:

Let X be a normal spherical variety and Y a G -orbit in X . Then the closure $\bar{Y}$ of Y in X is again a normal variety. $\square$

Corollary 2.13:

Let X be complex-symmetric with respect to G and G^0 semi-simple. If X is normal and algebraic and if G^0 acts algebraically, then the closure of each G^0 -orbit is again normal. $\square$

Before stating the Main theorem of this paragraph, we prove several technical Lemmas.

Lemma 2.14:

Let $X, Y := Q_{m_1} \times \dots \times Q_{m_n}$ be products of compact quadrics which are almost homogeneous with respect to

$$S := G_1 \times \dots \times G_n \cong SO(m_1, \mathbb{C}) \times \dots \times SO(m_n, \mathbb{C}) .$$

Denote the open S -orbits $Q_{(m_1)} \times \dots \times Q_{(m_n)}$ by Ω_X (resp. Ω_Y) .

Then every S -equivariant biholomorphic mapping

$$\varphi : \Omega_X \rightarrow \Omega_Y$$

extends to a unique S -equivariant biholomorphic mapping

$$\overline{\varphi} : X \rightarrow Y .$$

Proof:

There exist holomorphic mappings $f_i : \Omega_X \rightarrow Q_{(m_i)}$ such that for $x_i \in \Omega_X \cap Q_{m_i}$

$$\varphi(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)) .$$

Set $\varphi(x_1^0, \dots, x_n^0) := (y_1^0, \dots, y_n^0)$.

Then the S -equivariance of φ yields

$$f_i(g_1(x_1^0), \dots, g_n(x_n^0)) = g_i(y_i^0)$$

(Note that the S -equivariance implies that φ cannot exchange biholomorphic factors Q_{m_i} and Q_{m_j}) .

Therefore there exist G_i -equivariant biholomorphic mappings

$$\sigma_i : Q_{(m_i)} \rightarrow Q_{(m_i)}$$

such that $\varphi(x_1, \dots, x_n) = (\sigma_1(x_1), \dots, \sigma_n(x_n))$.

The Lemma is proved, if we show that there is a G_i -equivariant extension

$$\overline{\sigma_i} : Q_{m_i} \rightarrow Q_{m_i}, \text{ which is biholomorphic.}$$

But G_i acts on Q_{m_i} with two orbits, one of them being a divisor. The algebraicity of the isotropy-subgroups H_i and the finiteness of $N_{G_i}(H_i)/H_i$ imply that there is an extension of σ_i to a bimeromorphic mapping $\overline{\sigma_i} : X \rightarrow Y$.

But the set of indeterminacy of this mapping is G_i -invariant and has codimension at most two. Since the closed G_i -orbit is a divisor the set of indeterminacy of $\overline{\sigma_i}$ has to be empty and $\overline{\sigma_i}$ is holomorphic. The equivariance of $\overline{\sigma_i}$ implies that it is biholomorphic. $\square$

Lemma 2.15:

Let P_1, P_2 be parabolic subgroups of a semisimple group G .

Set $H := P_1 \cap P_2$. Then H has finitely many fixed points in G/P_1 .

Proof:

The parabolic subgroups P_1 (resp. P_2) contain Borel subgroups B_1 (resp. B_2). The intersection $B_1 \cap B_2$ contains a maximal torus Γ of G ([Hu], p. 173). Thus H is an algebraic subgroup of G containing a maximal torus. But a maximal torus has only a finite number of fixed points in G/P_1 (e.g. [Hu]). $\square$

Now consider the action of $G := SO(m, \mathbb{C})$ on the quadric Q_m .

For $x \in Q_{(m)}$, $H_x := \text{Iso}_x(G)$ is a connected subgroup of codimension $m-1$ in G . Thus $\mathfrak{h}_x := \text{Lie}(H_x)$ is an element of the

Grassmannian $Gr(m) := G\left(\frac{(m-1)(m-2)}{2}, \frac{m(m-1)}{2}\right)$. Therefore we can define a holomorphic mapping

$$\iota : Q_{(m)} \rightarrow Gr(m), \quad x \mapsto h_x \quad (\text{Note that } \tau \text{ is finite by Lemma 2.3})$$

which is G -equivariant via the adjoint representation of G on $\mathfrak{g} := \text{Lie}(G)$. This mapping is nothing but the normalizer-fibration of the open orbit $Q_{(m)} \subset Q_m$. It is known that the normalizer-fibration can be extended to an equivariant meromorphic mapping $\bar{\tau} : Q_m \rightarrow Gr(m)$ ([11-01], p. 85; in this case the existence of the meromorphic extension can be verified directly). Since the closed orbit is a divisor, $\bar{\tau}$ has to be holomorphic.

So we have proved

Lemma 2.16:

If $(x_n)_{n \in \mathbb{N}} \subset Q_{(m)}$ and $x_n \rightarrow y \in Q_m \setminus Q_{(m)}$, then $\tau(x_n) \rightarrow h_y \in Gr(m)$ (Note that h_y is perhaps not a subalgebra of $\mathfrak{g}$). $\square$

The following Lemma is based on the two preceding Lemmas and will be the central point in the proof of our Main Theorem.

So we have isolated it here.

Let $Y = Q_{m_1} \times \dots \times Q_{m_n}$ be a product of quadrics and consider the action of

$$S := SO(m_1, \mathbb{C}) \times \dots \times SO(m_n, \mathbb{C}) =: G_1 \times \dots \times G_n$$

on Y . Denote the open S -orbit in Y by Ω_Y .

Assume that there exists a compact normal space X on which S acts with an open orbit Ω_X and that there exists an S -equivariant biholomorphic mapping

$$\varphi : \Omega_Y \rightarrow \Omega_X, \text{ with } \text{Iso}_Y(S) = \text{Iso}_{\varphi(Y)}(S) \quad (y \in \Omega_Y) .$$

Let $\tau_i : Q_{m_i} \rightarrow \text{Gr}(m_i)$ be the holomorphic mappings introduced above, $\tau = (\tau_1, \dots, \tau_n)$ and

$$(x_k)_{k \in \mathbb{N}} = ((x_k^{(1)}, \dots, x_k^{(n)}))_{k \in \mathbb{N}}$$

be a sequence in Ω_Y with $x_k \rightarrow y \in Y \setminus \Omega_Y$.

By Lemma 2.16

$$\tau(x_k) \rightarrow (h_{1,y}, \dots, h_{n,y}) \in \bigtimes_{i=1}^n \text{Gr}(m_i) .$$

Now for a cluster point $x \in X$ of the sequence $(\varphi(x_k))_{k \in \mathbb{N}}$ the isotropy algebra $\text{iso}_x(S)$ fulfills the condition

$$\text{iso}_x(S) \supset (h_{1,y}, \dots, h_{n,y}) .$$

Lemma 2.17:

Assume that X has the following properties:

- (1) The compact G_i -orbits are compact quadrics Q_{m_i} .
- (2) The non-compact G_i -orbits are affine quadrics $Q_{(m_i)}$.
- (3) For each $x \in X$ we have

$$\text{Iso}_x(S) = \text{Iso}_x(G_1) \times \dots \times \text{Iso}_x(G_n) .$$

Then $\{x \mid \text{iso}_x(S) \supset (h_{1,y}, \dots, h_{n,y})\}$ is finite.

Proof:

Let $x \in X$ with the property

$$\text{Iso}_x(S) \supset (h_{1,y}, \dots, h_{n,y}) .$$

We will show that there are only finitely many points in $S(x)$ with the same property. This is enough since the number of S -orbits in X is finite.

If $\text{iso}_x(G_1)$ is reductive, then $\text{iso}_x(G_1) = h_{1,y}$, since both vector spaces have the same dimension. By Lemma 2.3 it follows that

$$\{\tilde{x} \in S(x) \mid \text{iso}_{\tilde{x}}(G_1) = \text{iso}_x(G_1) = h_{1,y}\}$$

consists of finitely many $(G_2 \cdots G_n)$ -orbits.

If $\text{iso}_x(G_1)$ is parabolic, then $\dim_{\mathbb{C}} h_{1,y} + 1 = \dim_{\mathbb{C}} \text{iso}_x(G_1)$. Now assume that $\text{iso}_{\tilde{x}}(G_1)$ also contains $h_{1,y}$, then there are two possibilities.

$$(i) \text{iso}_{\tilde{x}}(G_1) = \text{iso}_x(G_1) .$$

Then $\tilde{x} \in G_2 \cdots G_n(x)$, since $N_{G_1}(\text{Iso}_x(G_1)) = \text{Iso}_x(G_1)$.

$$(ii) \text{iso}_{\tilde{x}}(G_1) \cap \text{iso}_x(G_1) = h_{1,y} .$$

Then $h_{1,y}$ is a subalgebra of $\mathfrak{so}(m_1, \mathbb{C})$ and the corresponding Lie group is closed, algebraic and contains a maximal torus.

By Lemma 2.15 there exists only finitely many $\tilde{x} \in G_1(x)$ with

$$h_{1,y} \subset \text{iso}_{\tilde{x}}(G_1) .$$

Consequently,

$$\{\tilde{x} \in S(x) \mid \text{iso}_{\tilde{x}}(G_1) \supset h_{1,y}\}$$

consists of finitely many $(G_2 \cdots G_n)$ -orbits.

Now the same arguments can be applied to the other factors.

Thus it follows that

$$\{\tilde{X} \in S(x) \mid \text{Iso}_{\tilde{X}}(G_1) \supset h_{1,y}, \dots, \text{Iso}_{\tilde{X}}(G_n) \supset h_{n,y}\}$$

is a finite set. $\square$

Remark:

If X is complex-symmetric with respect to G with G^0 semi-simple, then we will show that there exists an S -equivariant mapping $\varphi : \Omega_Y \rightarrow \Omega_X$ as above.

Moreover, the assumptions of Lemma 2.17 are fulfilled (Lemma 2.1; Lemma 2.9). Thus Lemma 2.17 can be applied in the Main Theorem which now can be stated.

Theorem 2.18:

Let (X, G) be complex-symmetric, G^0 semisimple and let the open G^0 -orbit Ω in X be Stein.

Assume

- (a) X is smooth or
- (b) X is normal, algebraic and the action of G^0 on X is algebraic.

Then Ω is the product

$$\Omega = Q_{(m_1)} \times \dots \times Q_{(m_n)}$$

of affine quadrics, X is correspondingly the product

$$X = Q_{m_1} \times \dots \times Q_{m_n}$$

of irreducible compact quadrics and therefore a homogeneous space for the group

$$SO(m_1 + 1, \mathbb{C}) \times \dots \times SO(m_n + 1, \mathbb{C}) .$$

Proof:

Consider the decomposition of

$$g \cong so(m_1, \mathbb{C}) \oplus \dots \oplus so(m_n, \mathbb{C})$$

in normal factors which are stable under the symmetries of Ω and which are minimal with this property.

We proceed by induction on n (The case $n = 1$ is Lemma 2.9).

Let $G^0 = G_1 \cdot G_2 \cdot \dots \cdot G_n$. Since Ω is Stein there exists a G^0 -orbit E_i of codimension one in X such that the G_i -orbits in E_i are compact and the G_j -orbits in E_i are Stein for $j \neq i$ (Lemma 2.10).

- (a) If X is smooth, then $\overline{E_i}$ is smooth by Lemma 2.11 and we can apply induction on $\overline{E_i}$.
- (b) If X is normal and algebraic and the G^0 -action on X is algebraic, then $\overline{E_i}$ is normal by Corollary 2.13 and we can apply induction on $\overline{E_i}$.

Thus in both cases we have that E_i is a product of Q_{m_i-1} and affine quadrics and that $\overline{E_i}$ is a product of Q_{m_i-1} and compact quadrics.

In order to simplify our notation the open orbit of $SO(m_1, \mathbb{C}) \times \dots \times SO(m_n, \mathbb{C})$ in $Q_{m_1} \times \dots \times Q_{m_n} =: \tilde{X}$ is denoted by $\tilde{\Omega}$, the codimension one orbits by A_i ($i = 1, \dots, n$) and the closed orbit in $\tilde{X}$ by $\tilde{A}$.

Now our inductive argument is organized as follows.

Step 1:

There is a natural equivariant biholomorphic mapping

$$\varphi : \tilde{\Omega} \rightarrow \Omega .$$

Step 2:

The mapping φ can be extended to a bimeromorphic mapping

$$\bar{\varphi} : \tilde{X} \rightarrow X .$$

The restriction

$$\bar{\varphi} : \tilde{\Omega} \cup \left(\bigcup_{i=1}^n A_i \right) \rightarrow \Omega \cup \left(\bigcup_{i=1}^n E_i \right)$$

is indeed biholomorphic.

Step 3:

The meromorphic extension

$$\bar{\varphi} : \tilde{X} \rightarrow X$$

is biholomorphic.

Step 1:

By replacing G^0 by a finite covering if necessary we may assume that

$$G^0 = G_1 \times \dots \times G_n \cong SO(m_1, \mathbb{C}) \times \dots \times SO(m_n, \mathbb{C}) .$$

Then the open orbit $\Omega \subset X$ is the form G^0/H and the open orbit $\tilde{\Omega}$ in $\tilde{X}$ is of the form G^0/H^0 since $\tilde{\Omega}$ is a product of affine quadrics. We have to prove that $H = H^0$.

For an affine quadric G_i/H_i we have $N_{G_i}(H_i)/H_i \cong \mathbb{Z}_2$, since a symmetry on the affine quadric has exactly two fixed points.

Therefore $N_{G^0}(H^0)/H^0 \cong (\mathbb{Z}_2)^n$, and we have a well-defined right action of this group on $\tilde{\Omega} = G^0/H^0$ by

$$gH^0 \rightarrow gnH^0 \quad (g \in G^0, n \in N_{G^0}(H^0)).$$

The corresponding action of $N_{G_i}(H_i)/H_i$ on $Q_{(m_i)}$ can be extended meromorphically on Q_{m_i} since the action is algebraic. This extension is indeed holomorphic since the exceptional orbit is a divisor.

As shown in Lemma 2.14 the normalizer fibration

$$\varphi : Q_{(m_i)} = G_i/H_i \rightarrow G_i/N_{G_i}(H_i) \cong \mathbb{P}_{m_i-1} \setminus Q_{m_i-1} \quad ([A])$$

of the open orbit can be extended to a holomorphic mapping

$$\varphi : Q_{m_i} \rightarrow \mathbb{P}_{m_i-1}$$

and the restriction to the exceptional orbits is biholomorphic and equivariant.

Thus the extension of the action of $N_{G_i}(H_i)/H_i$ on G_i/H_i is trivial on the compact G_i -orbit.

Therefore the action of $N_{G^0}(H^0)/H^0 = N_{G_1}(H_1)/H_1 \times \dots \times N_{G_n}(H_n)/H_n$ on $\tilde{\Omega}$ can be extended to an action on $\tilde{X}$ which is trivial on the compact G^0 -orbit.

Now $\Gamma := H/H^0$ is a subgroup of $N_{G^0}(H^0)/H^0$. Thus we have a holomorphic (and algebraic) action of a finite group on $\tilde{X}$ and we can consider the quotient

$$Z := \Gamma \backslash \tilde{X}$$

by the right action of Γ .

The quotient Z is algebraic and normal (since $\tilde{X}$ is normal), and the open orbit $\Omega_Z := \Gamma \backslash \tilde{\Omega}$ is isomorphic to Ω because the mapping

$$G^0/H \rightarrow \Omega_Z, \quad gH \mapsto [gH^0] \quad (\text{equivalence class})$$

is well-defined, injective and surjective.

The finiteness of $N_{G^0}(H^0)/H^0$ and the algebraicity of Z imply that the biholomorphic mapping

$$G^0/H \rightarrow \Omega_Z$$

can be extended to a bimeromorphic mapping

$$\varphi : X \dashrightarrow Z.$$

Now for each $i \in \{1, \dots, n\}$ there exists exactly one G^0 -orbit E_i of codimension one in X (resp. in Z) such that

$$\overline{G_i(X)} \setminus G_i(x) \subset E_i \quad \forall x \in \Omega \quad (\text{resp. } \overline{G_i(X)} \setminus G_i(x) \subset A_i \quad \forall x \in \Omega_Z).$$

This follows from Lemma 2.7 for $x \in \Omega$ and from the product structure of $\tilde{X}$ for $x \in \Omega_Z$.

Since the restriction of φ (resp. φ^{-1}) to a G^0 -orbit of codimension one is holomorphic, the equivariance of φ implies that for each G^0 -orbit $E_X \subset X$ of codimension one there exists a G^0 -orbit $E_Z \subset Z$ of codimension one with

$$\varphi(E_X) = E_Z \quad (\text{resp. } \varphi^{-1}(E_Z) = E_X).$$

Thus $\varphi|_{E_X} : E_X \rightarrow E_Z$ is biholomorphic.

Let $E_i := G_1/H_1 \times \dots \times G_{i-1}/H_{i-1} \times G_i/P_i \times G_{i+1}/H_{i+1} \times \dots \times G_n/H_n$ be a G^0 -orbit of codimension one in $\tilde{X}$ where G_i has compact orbits. Since Γ is a subgroup of $N_{G^0}(H^0)/H^0 = N_{G_1}(H_1)/H_1 \times \dots \times N_{G_n}(H_n)/H_n$ and $N_{G_i}(H_i)/H_i$ acts trivially on G_i/P_i , we have

$$\Gamma \backslash E_i \cong G_i/P_i \times \Gamma'_i \backslash (G_1/H_1 \times \dots \times G_{i-1}/H_{i-1} \times G_{i+1}/H_{i+1} \times \dots \times G_n/H_n)$$

where Γ'_i is the image of Γ under the homomorphism induced by the projection

$$\pi_i : G_1 \times \dots \times G_n \rightarrow G_1 \times \dots \times G_{i-1} \times G_{i+1} \times \dots \times G_n.$$

As shown above the G^0 -orbit $\Gamma \backslash E_i \subset Z$ is biholomorphic to a G^0 -orbit $\tilde{E}_i$ of codimension one in X . By induction

$$\tilde{E}_i \cong G_1/H_1 \times \dots \times G_{i-1}/H_{i-1} \times G_i/P_i \times G_{i+1}/H_{i+1} \times \dots \times G_n/H_n.$$

Thus for each $i \in \{1, \dots, n\}$ the group Γ'_i has to be trivial. Since $n \geq 2$ this is only possible, if the group Γ is trivial. Thus $H = H^0$ and the open orbit Ω in X is biholomorphic to a product of affine quadrics

$$\tilde{\Omega} = G_1/H_1 \times \dots \times G_n/H_n.$$

Step 2:

It follows either from the algebraicity of the G -action or from the fact that the normalizer-fibration of the open orbit is finite by Lemma 2.3 that the biholomorphic equivariant mapping

$$\varphi : \tilde{\Omega} \rightarrow \Omega$$

can be extended to a bimeromorphic mapping

$$\bar{\varphi} : \tilde{X} \rightarrow X.$$

Let A_i (resp. E_i) be the G^0 -orbits of codimension one in $\tilde{X}$ (resp. X) where $G_i \cong SO(m_i, \mathbb{C})$ has compact orbits. Then $\bar{\varphi}(\tilde{\Omega} \cup A_i)$ is holomorphic since $\tilde{X}$ is normal.

By Lemma 2.7 for each $y \in E_i$ there exists exactly one G_i -orbit $G_i(x)$ in Ω containing y in its closure. Moreover, $\overline{G_i(x)}$ is smooth and complex-symmetric (Lemma 2.7) and thus isomorphic to Q_{m_i} by Lemma 2.9.

Since $\tilde{X}$ is a product of quadrics each $\tilde{y} \in A_i$ lies in the closure of exactly one G_i -orbit $G_i(\tilde{x})$ in $\tilde{\Omega}$.

The equivariance of φ implies that

$$\bar{\varphi}(\tilde{\Omega} \cup A_i) \subset (\Omega \cup E_i)$$

and that $\bar{\varphi}|(\tilde{\Omega} \cup A_i)$ is surjective.

It is also injective.

Otherwise for $y \in E_i$ there exist two disjoint G_i -orbits in Ω containing y in their closure. This yields a contradiction to Lemma 2.8. Thus

$$\bar{\varphi}|(\tilde{\Omega} \cup A_i) : \tilde{\Omega} \cup A_i \rightarrow \Omega \cup E_i$$

is biholomorphic.

□

Step 3:

By induction we know that the closure of each G^0 -orbit of codimension one in X is a product of compact quadrics.

The mapping $\bar{\varphi}$ defined in Step 2 induces a biholomorphic equivariant mapping

$$\bar{\varphi} : A_i \rightarrow E_i .$$

So by Lemma 2.14 this mapping can be extended equivariantly to a biholomorphic mapping

$$\tilde{\varphi}_i : \bar{A}_i \rightarrow E_i .$$

It is now possible to define an extension

$$\begin{aligned} \tilde{\varphi} : \tilde{X} \rightarrow X \text{ of } \bar{\varphi} \text{ by} \\ \tilde{\varphi}(\tilde{x}) = \tilde{\varphi}_i(\tilde{x}) \text{ if } \tilde{x} \in \bar{A}_i . \end{aligned}$$

Note that we have to prove that this mapping is well-defined and continuous. The bijectivity then follows automatically from our construction.

If we prove that the meromorphic mapping $\bar{\varphi}$ introduced in Step 2 is holomorphic then $\bar{\varphi}$ and $\tilde{\varphi}$ coincide. Thus we have found a biholomorphic equivariant extension of φ , and the Theorem is proved by induction.

In order to prove that $\bar{\varphi}$ is holomorphic it is enough to show that for each $\tilde{y} \in \tilde{X}$ the irreducible analytic set $\bar{\varphi}(\{\tilde{y}\})$ is finite.

Now let $\tilde{y} \in \tilde{X} \setminus (\tilde{X} \cup \bigcup_{i=1}^n A_i)$ and $(h_1, \tilde{y}, \dots, h_n, \tilde{y}) := \tau(\tilde{y})$ (cf. Lemma 2.17).

Then a point $y \in \bar{\varphi}(\tilde{y})$ satisfies the condition

$$\text{iso}_y(G^0) \supset (h_1, \tilde{y}, \dots, h_n, \tilde{y}) .$$

But this set is finite by Lemma 2.17 (The assumptions of this Lemma are fulfilled; cf. the Remark following Lemma 2.17). $\square$

Remark:

The basic idea of our proof can be summarized as follows.

Let $\tilde{X}$ be the product of compact quadrics and X the complex-symmetric space. We know that the normalizer-fibration $\tilde{\tau} : \tilde{X} \rightarrow \tilde{Z}$ is holomorphic and finite. Let τ be the meromorphic extension of the normalizer fibration of X . Then we have the diagram

$$\begin{array}{ccc} \bar{\varphi} : \tilde{X} & \xrightarrow{\quad} & X \\ & \searrow \tilde{\tau} & \downarrow \tau \\ & & \tilde{Z} \end{array}$$

Now Lemma 2.17 implies that τ is holomorphic and finite. Thus $\bar{\varphi}$ is holomorphic and it follows that $\bar{\varphi}$ is biholomorphic since $\bar{\varphi}|_{\Omega}$ is biholomorphic.

Corollary 2.19:

Each smooth complex-symmetric space (X, G) with G^0 semisimple is homogeneous and therefore Hermitian symmetric. $\square$

Corollary 2.20:

Let X be a normal algebraic space which is complex-symmetric with respect to G . Assume that G acts algebraically and that G^0 is semisimple.

Then X is smooth and therefore homogeneous. $\square$

§ 5. Complex-symmetric Spaces with G^0 Reductive

In § 2 we have considered compact manifolds X which are complex-symmetric with respect to a semisimple group G . We have shown that

$$X \cong Y \times Q_{m_1} \times \dots \times Q_{m_n},$$

where Y is a Hermitian symmetric space of the compact type and each Q_{m_i} is an irreducible compact quadric on which the orthogonal group $SO(m_i, \mathbb{C})$ acts with two orbits, the open orbit being an affine quadric $Q_{(m_i)}$.

Our aim here is to give a description of all manifolds X which are complex-symmetric with respect to a reductive group.

If we assume that (X, G) is complex-symmetric and that G^0 is algebraic and reductive then

$$G^0 = S \cdot T \quad (S \text{ and } T \text{ normal factors with finite intersection})$$

where S is semisimple and T is an algebraic torus (i.e. an algebraic group isomorphic to $(\mathbb{C}^*)^k$).

Of course a symmetry s induces an automorphism σ of the Lie algebra $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{t}$ with $\sigma(\mathfrak{s}) = \mathfrak{s}$ and $\sigma(\mathfrak{t}) = \mathfrak{t}$.

Therefore for $x \in X$ the symmetry s_x stabilizes $S(x)$ and $T(x)$, and some of the results of the preceding paragraph can be applied.

Now before stating the Main Result of this paragraph we recall the notion of a torus embedding from the introduction.

Definition:

A compact torus embedding is an algebraic compactification X of an algebraic torus $T \cong (\mathbb{C}^*)^k$ such that the natural action of T on T given by the group action extends to an algebraic action of T on X .

A torus embedding X is called complex-symmetric, if X is complex-symmetric with respect to G with $G^0 = T$.

Now we can state our Main Result.

Theorem 3.1:

Let X be a compact complex projective algebraic manifold. Assume that X is complex-symmetric with respect to a complex Lie group G with G^0 reductive and that the action of G is linear algebraic.

Then X is a product of a smooth complex-symmetric torus embedding and a complex-symmetric space with a semisimple automorphism group.

Remarks:

- (1) In contrast to the semisimple case we need the algebraicity of the G -action on X for our proof. It is necessary for instance to assure that the isotropies of the T -action on X are connected and algebraic.
- (2) It is perhaps possible that the above Theorem remains valid if we replace "smooth" by "normal". The variety X is spherical and by Proposition 2.12 there will be an inductive situation. But the arguments will necessarily become much more complicated.

- (3) Combining the above Theorem with Theorem 2.18 and the results of part II we obtain a complete classification of smooth projective complex-symmetric varieties (X, G) with an algebraic action of the reductive group G^0 .
- (4) For details concerning torus embeddings we refer to part II, § 1.1 and § 1.2.

The proof of Theorem 3.1:

We proceed by induction on the complex dimension of X . The case of dimension one is trivial.

Let $G^0 = T \cdot S$ and $k := \dim_{\mathbb{C}} T$.

Since G^0 is acting effectively, the action of T (resp. S) on X is effective.

Now consider a Lie algebra automorphism σ induced by a symmetry s . Since

$$\sigma(\mathfrak{s}) = \mathfrak{s} \quad \text{and} \quad \sigma(\mathfrak{t}) = \mathfrak{t},$$

there exists a decomposition

$$\mathfrak{s} = \mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_n$$

in ideals which are stable under each automorphism induced by a symmetry and which are minimal for the symmetries of the open G^0 -orbit Ω (cf. Lemma 2.2).

The orbits of the corresponding normal subgroups S_i in X are either compact or Stein (Lemma 2.1).

Now if for $x \in \Omega$ the S_1 -orbit $S_1(x)$ is compact, by Lemma 2.4

$$X = S_1(x) \times Y$$

where Y is again complex-symmetric with respect to a reductive group.

By induction Y is a product. Thus X is a product. So in this case the Theorem is proved immediately.

Therefore we can restrict ourselves to the case where the open G^0 -orbit Ω is Stein.

Since the proof is a bit technical it is divided into several steps.

First we give an

Outline of the proof:

Step 1:

Let E be a G^0 -orbit of codimension one in X . Then $\bar{E}$ is smooth. By induction the closure of each G^0 -orbit is smooth.

Step 2:

There exists a G^0 -orbit of codimension one where the T -orbits are not maximal-dimensional.

By induction and by linearization in fixed points we then show that the S_1 -orbits in Ω are affine quadrics.

Step 3:

For each factor S_i of S there exists a G^0 -orbit of codimension one where S_i has compact orbits.

By Lemma 2.8 the closure of each S_i -orbit in Ω is contained in this G^0 -orbit and the closure $\overline{S_i(x)}$ of each S_i -orbit $S_i(x) \subset \Omega$ is smooth.

Step 4:

For each $x \in X$ the closures $\overline{T(x)}$ (resp. $\overline{S(x)}$) of the T -orbits (resp. S -orbits) are smooth.

So by Theorem 2.18 we know that for $x \in \Omega$

$\tilde{Q} := S(x)$ is a product of affine quadrics and

$Q := \overline{S(x)}$ is a product of compact quadrics.

Step 5:

For each $x \in X$ we have

$T(x) \cap S(x) = \{x\}$, i.e. for $x \in \Omega$ we have a biholomorphic

mapping

$$\varphi : S(x) \times T(x) \cong \tilde{Q} \times T(x) \rightarrow \Omega.$$

Step 6:

The mapping φ extends to a biholomorphic mapping

$$\bar{\varphi} : Q \times T(x) \rightarrow A := \{y \in X \mid T(y) \cong (\mathbb{C}^*)^k\}.$$

The set A consists of those points where the T -orbits are maximal-dimensional.

Step 7:

The mapping φ extends to a biholomorphic mapping

$$\bar{\varphi} : \tilde{Q} \times \overline{T(x)} \rightarrow B := \{y \in X \mid S(y) \cong \tilde{Q}\}.$$

The set B consists of those points where the S -orbits are maximal-dimensional.

In order to show that φ extends to a biholomorphic mapping

$$\bar{\varphi} : Q \times \overline{T(x)} \rightarrow X$$

we need

Step 8:

The set $X \setminus (A \cup B)$ is contained as well in $(\overline{A \setminus \Omega})$ as in $(\overline{B \setminus \Omega})$.

It follows from this Step that each $y \in X \setminus (A \cup B)$ is contained in the closure of one S -orbit $S(\tilde{y})$ in B .

This enables us to prove that there is a biholomorphic mapping

$\bar{\varphi} : Q \times \overline{T(x)} \rightarrow X$ by showing that the mappings $\pi_1 \circ \bar{\varphi}^{-1}$ (resp. $\pi_2 \circ \bar{\varphi}^{-1}$) are holomorphic. Here π_1 (resp. π_2) denotes the projection from $Q \times \overline{T(x)}$ onto Q (resp. $\overline{T(x)}$).

Note that it follows automatically that the closures of all maximal-dimensional T -orbits are T -equivariantly biholomorphic.

Step 1:

The closure of each G^0 -orbit in X is smooth.

Proof:

Since the open G^0 -orbit is assumed to be Stein each orbit of codimension bigger than one is contained in the closure of a G^0 -orbit of codimension one.

So it is enough to prove the statement for the G^0 -orbits of codimension one since we can then apply induction to the lower-dimensional G^0 -orbits. Note that the closures of T -orbits in a smooth torus embedding are again smooth (cf. part II; § 1.2).

Now let E be a G^0 -orbit of codimension one. Then there are two possibilities for E :

- (1) The T -orbits in E are $(k-1)$ -dimensional, where $k := \dim_{\mathbb{C}} T$.
- (2) The T -orbits in E are k -dimensional.

This is due to the fact the the T -orbits in Ω are k -dimensional and that the S -orbits in Ω are of maximal dimension among all S -orbits in X .

- (1) In this case an algebraic one-parameter-subgroup T_1 acts trivially on E and thus on $\bar{E}$.

Since E is irreducible and the set of fixed points of a reductive group in a manifold is a smooth analytic subset (by faithful linearization), the set E is a connected component of this fixed point set and thus smooth.

- (2) In this case by Lemma 2.8 there exists a factor S_i of S such that $\overline{S_i(x)} \setminus S_i(x) \subset E$ for all $x \in \Omega$. Moreover, $\overline{S_i(x)}$ is smooth and complex-symmetric, thus $S_i(x)$ is an affine quadric $Q_{(m_i)}$ and $\overline{S_i(x)}$ is a compact quadric Q_{m_i} .

Now the same arguments as in the proof of Lemma 2.4 show that all S_i -orbits in E are compact and equivariantly biholomorphic. As in Lemma 2.11 it follows that $\bar{E}$ is a connected component of the set where the isotropies of the K_1 -action are conjugates of a fixed isotropy (K_1 a maximal compact subgroup of S_1). Thus $\bar{E}$ is smooth.

Remark:

Let E be a G^0 -orbit of codimension one where an algebraic one-parameter-subgroup T_1 acts trivially and $x \in E$. Since the action of T on X is effective, faithful linearization of the algebraic group $\text{Iso}_x(T)$ yields a faithful representation

$$\rho : \text{Iso}_x(T) \rightarrow \mathbb{C}^*.$$

By Lemma 2.6 the group $\text{Iso}_x(T)$ is connected, i.e.

$$\text{Iso}_x(T) = T_1 \quad \forall x \in E. \quad \square$$

Before we proceed with Step 2 we make a general remark.

By our assumption the manifold X is projective algebraic and the group G^0 is acting linear algebraically.

Consider the S -action (resp. T -action) on X . Then the geometric quotient of the open G^0 -orbit Ω with respect to the S -action (resp. T -action) exists.

Let $\varphi_S : \Omega \rightarrow \Omega_{Y_S}$ (resp. $\varphi_T : \Omega \rightarrow \Omega_{Y_T}$) be the (algebraic) mappings onto the geometric quotient. Then it is known that there exist equivariant embeddings

$$\Omega_{Y_S} \hookrightarrow Y_S \quad \text{and} \quad \Omega_{Y_T} \hookrightarrow Y_T$$

(Ω_{Y_S} Zariski-open in Y_S) such that the mappings φ_S and φ_T introduced above extend equivariantly to meromorphic mappings

$$\varphi_S : X \dashrightarrow Y_S \quad \text{and} \quad \varphi_T : X \dashrightarrow Y_T.$$

These mappings will play an important role in our proof of Theorem 3.1.

Step 2:

There exists a G^0 -orbit of codimension one where the generic T -orbits of the torus T are not maximal-dimensional.

Proof:

Let $E_1, \dots, E_S$ be all G^0 -orbits of codimension one and $F := \Omega \cup \left(\bigcup_{j=1}^S E_j \right)$.

Proceeding by contradiction we assume that all T -orbits in F are maximal-dimensional.

Consider the geometric quotient

$$\varphi_S : \Omega \rightarrow \Omega_{Y_S}.$$

Then $\Omega_{Y_S} \cong (\mathbb{C}^*)^k$ is Stein.

Since φ_S can be extended meromorphically to all of X and X is normal it follows that $\varphi_S|_F$ is holomorphic. Since the T -orbits in F are maximal-dimensional, by Lemma 2.8 there exists a factor S_i of S with $\overline{S_i(x)} \setminus S_i(x) \subset E_i$ for $E_i \subset F$ and $x \in \Omega$. The G -equivariance of φ_S implies

$$\varphi_S(F) = \Omega_{Y_S}.$$

Thus there exists a non-constant holomorphic function on $\varphi_S(F)$ which can be lifted to a non-constant holomorphic function on F . But $X \setminus F$ is two-codimensional by our construction. Since X is compact and normal this yields a contradiction. $\square$

By Step 2 there exists a G^0 -orbit E of codimension one where the T -orbits are $(k-1)$ -dimensional ($k := \dim_{\mathbb{C}} T$). By faithful linearization there exists an algebraic one-parameter-subgroup T_1 of T such that $y \in E$ is contained in the closure of the T_1 -orbit $T_1(x)$ in Ω (T_1 is the ineffectivity of the T -action on E).

Therefore

$$\text{Iso}_y(S) > \text{Iso}_x(S) ,$$

and both groups have the same dimension.

Therefore we have a finite fibration

$$S(x) = S/\text{Iso}_x(S) \rightarrow S/\text{Iso}_y(S) = S(y) .$$

By induction the base of this fibration is a product of affine quadrics and therefore simply connected. So $S(x)$ is a product of affine quadrics. Especially, we have.

$$S_i(x) \cap S_j(x) = \{x\} \quad (i \neq j ; x \in \Omega) .$$

Step 3:

For each factor S_i of S there exists a unique G^0 -orbit of codimension one where S_i has compact orbits.

Proof:

This is essentially the same argumentation as in Lemma 2.10.

The uniqueness follows directly from the fact that the non-compact S_i -orbits have one end (cf. Lemma 2.10), so we only have to prove the existence.

Let $E_1, \dots, E_{S_S}$ be the G^0 -orbits of codimension one and
 $F := \Omega \cup \left(\bigcup_{j=1}^S E_j \right) .$

Proceeding by contradiction assume that all S_i -orbits in F are non-compact. Then they are all biholomorphic to an affine quadric S_i/H_i (by induction and by the above Remark).

For $x \in \Omega$ the isotropy subalgebra $\text{Iso}_x(S_i)$ is an element of a certain Grasmannian Z .

Consider the mapping

$$\tau : \Omega \rightarrow Z, x \mapsto \text{Iso}_x(S_i).$$

This yields an S_i -equivariant mapping and $\tau(\Omega)$ is an algebraic S_i -orbit which is isomorphic to $S_i/N_{S_i}(H_i)$. Lemma 2.3 implies that $\tau(\Omega)$ is Stein.

The same arguments as in Lemma 2.10 imply that τ can be extended continuously to a mapping

$$\tau : F \rightarrow \tau(\Omega).$$

Here the assumption that the non-compact S_i -orbits are affine quadrics is essential.

Since $\tau(\Omega)$ is Stein there exists a non-constant holomorphic function f on $\tau(\Omega)$. Then $f \circ \tau$ is a non-constant holomorphic function on F . Since $X \setminus F$ is two-codimensional we obtain a contradiction. $\square$

Since the following Lemmas are important for the arguments in the next Steps we have isolated them here.

Lemma 3.2:

Let A be the set of those points in X where the T -orbits are maximal-dimensional. Let

$$\varphi_S : X \rightarrow Y_S$$

be the meromorphic extension of the mapping on the geometric quotient by the S -orbits.

Then $\varphi_S|A$ is holomorphic and $\varphi_S(A) = (\mathbb{C}^*)^k$.

Proof:

Let $E_1, \dots, E_s$ be all G^0 -orbits of codimension one, let $E_1, \dots, E_t$ be those orbits of codimension one where the T -orbits are maximal-dimensional and let $F := \Omega \cup \left(\bigcup_{j=1}^t E_j \right)$.

Then F is dense in A , and $A \setminus F$ has codimension two in A . Since φ_S can be extended meromorphically it follows that $\varphi_S|F$ is holomorphic.

Since the T -orbits in F are maximal-dimensional, for each $E_i \subset F$ there exists a factor $S_i \triangleleft S$ such that

$$\overline{S_i(x)} \setminus S_i(x) \subset E_i \quad \forall x \in \Omega \quad \text{by Lemma 2.8.}$$

Thus

$$\varphi_S(F) = \varphi_S(\Omega) \cong (\mathbb{C}^*)^k \hookrightarrow \mathbb{C}^N.$$

Since $A \setminus F$ has codimension two and the mapping is given by holomorphic functions, it follows that $\varphi_S|F$ is holomorphically extendable to all of A . The T -equivariance of φ_S implies

$$\varphi_S(A) = \varphi_S(\Omega). \quad \square$$

Lemma 3.3:

Let B be the set of those points in X where the S -orbits are maximal-dimensional. Let

$$\varphi_T : X \rightarrow Y_T$$

be the meromorphic extension of the mapping on the geometric quotient by the T -orbits. Then $\varphi_T|B$ is holomorphic.

Proof:

By the Remark following Step 2 all S-orbits of maximal dimension are biholomorphic to a product $\tilde{Q}$ of affine quadrics.

Let $E_1, \dots, E_p$ be those G^0 -orbits of codimension one where the S-orbits are maximal-dimensional and let $F := \Omega \cup \left(\bigcup_{j=1}^p E_j \right)$.

Then F is dense in B and $B \setminus F$ has codimension two in B .

By faithful linearization of the T-isotropy it follows that for each $E_i \subset F$ there exists a one-parameter subgroup $T_i < T$ such that $\overline{T_i(x)} \setminus T_i(x) \subset E_i \quad \forall x \in \Omega$.

Since $\varphi_T|_F$ is holomorphic we have

$$\varphi_T(F) = \varphi_T(\Omega) = : \hat{Q} \hookrightarrow \mathbb{C}^M$$

where $\tilde{Q}$ is a finite covering of $\hat{Q}$, since $S(x) \cap T(x)$ is finite for $x \in \Omega$.

Since $B \setminus F$ has codimension two and the mapping is given by holomorphic functions it follows that $\varphi_S|_F$ is holomorphically extendable to all of B and that

$$\varphi_T(B) = \varphi_T(\Omega) \quad . \quad \square$$

Step 4:

For each $x \in X$ the analytic sets $\overline{T(x)}$ and $\overline{S(x)}$ are smooth.

Proof:

We have shown in Step 1 that the closure of each G^0 -orbit is smooth.

By induction each lower-dimensional G^0 -orbit is a product of a smooth torus embedding and of compact quadrics. Since the closures of the T-orbits in a smooth torus embedding are again smooth (part II, Prop. 13) and the closures of the S-orbits in $X \setminus \Omega$ are smooth by induction the statement is clear for $x \in X \setminus \Omega$.

So we assume $x \in \Omega$.

(1) We first prove the statement for the T-orbits.

Note that for $H := \text{Iso}_X(S)$ ($x \in \Omega$) the finiteness of $N_S(H)/H$ implies that each connected component of the smooth set of fixed points of H has at most dimension $k := \dim_{\mathbb{C}} T$.

Now if $T(x)$ is a T-orbit in Ω , $\overline{T(x)}$ is an irreducible analytic subset contained in the smooth set of fixed points of H . Since both sets have the same dimension the set $\overline{T(x)}$ is smooth.

(2) Now we consider the closures $\overline{S(x)}$ of the S-orbits $S(x) \subset \Omega$. First we will show that $\overline{S(x)}$ is contained in the set A of those points where T has maximal-dimensional orbits. Since $S(x)$ is a product of affine quadrics it is spherical (cf. § 2). The algebraicity of the S-action on X implies that $S(x)$ is Zariski-open in its closure $\overline{S(x)}$. Therefore $\overline{S(x)}$ is an "embedding" of $S(x)$ in the sense of [L-V] and S acts with a finite number of orbits on $\overline{S(x)}$ ([Br], p. 196). The arguments in Lemma 2.5 (cf. the general Lemma on p. 31) show that $\overline{S(x)}$ contains a unique compact S-orbit.

Now $\overline{S(x)}$ contains $\overline{S_1(x)} \subset A$ (Lemma 2.8). Let $y \in \overline{S_1(x)} \setminus S_1(x)$. Since $\overline{S(y)}$ is a product by induction it follows that $\overline{S(y)} \subset A$ and that the unique compact S-orbit in $\overline{S(x)}$ is contained in A .

Proceeding by contradiction we assume $\overline{S(x)} \not\subset A$. Let $\tilde{y} \in \overline{S(x)} \setminus A$. Then $\overline{S(\tilde{y})} \subset \overline{S(x)}$ and $\overline{S(\tilde{y})}$ contains a compact S-orbit. But since $\tilde{y} \notin A$, we have that $\text{Iso}_{y'}(T)$ is positive-dimensional for all $y' \in \overline{S(\tilde{y})}$. Thus there exist at least two compact S-orbits in $\overline{S(x)}$. Contradiction.

Since $\overline{S(x)} \subset A$ and $\varphi_S|_A$ is holomorphic by Lemma 3.2 the set $\overline{S(x)}$ is contained in an appropriate fiber of $\varphi_S|_A$ as an irreducible component of this fiber. The generic fibers of $\varphi_S|_A$ are smooth because A is smooth. But all fibers of $\varphi_S|_A$ are T -equivariantly biholomorphic. Therefore all fibers are smooth. Thus $\overline{S(x)}$ is smooth for each $x \in \Omega$. $\square$

Step 5:

For $x \in X$

$$S(x) \cap T(x) = \{x\},$$

and there is a biholomorphic mapping

$$\varphi : S(x_0) \times T(x_0) \cong \tilde{Q} \times T \rightarrow \Omega \quad (x_0 \in \Omega)$$

where Ω is the open G^0 -orbit in X and $\tilde{Q}$ is a product of affine quadrics.

Proof:

By induction and by Step 1 the first statement is clear for $x \in X \setminus \Omega$.

So let $x \in \Omega$. Without loss of generality we may assume that

$$G^0 = S \times T$$

and that S and T act effectively on X .

Define $H := \text{Iso}_X(S)$.

Since $S(x)$ is a product of affine quadrics we have $H = H^0$.

Let $N := N_S(H^0)$. Then

$$N(x) = : \{x_1 = x, x_2, \dots, x_{2n}\}, \text{ since } N/H^0 \cong \mathbb{Z}_2^n.$$

Proceeding by contradiction we assume that $S(x) \cap T(x) \supset \{x, y\}$. Since T and S are normal factors

$$\text{Iso}_y(S) = \text{Iso}_x(S) .$$

Thus there exist $s_1 \in N$ and $t_1 \in T$ such that

$$s_1(x) = y = t_1(x) .$$

Let E be a G^0 -orbit of codimension one where the torus T has one-dimensional isotropy. By induction the S -orbits in E are products of affine quadrics.

We can find a point $z \in E$ such that

$$z \in \overline{T(x)}$$

by faithful linearization of the T -isotropy in E .

It follows that $\text{Iso}_2(S) = \text{Iso}_x(S) = H$ and that $s_1(z) \neq z$ since $s_1 \in N \setminus H$ by construction.

Now let $(t^{(v)})_{v \in \mathbb{N}}$ be a sequence in T with

$$\lim_{v \rightarrow \infty} t^{(v)}(x) = z .$$

Then

$$\begin{aligned} s_1(z) &= \lim_{v \rightarrow \infty} s_1(t^{(v)}(x)) = \lim_{v \rightarrow \infty} t^{(v)} s_1(x) \quad (\text{since } T \text{ is central}) \\ &= \lim_{v \rightarrow \infty} t^{(v)} t_1(x) \quad (\text{by definition of } t_1 \text{ and } s_1) \\ &= \lim_{v \rightarrow \infty} t_1 t^{(v)}(x) \quad (T \text{ is commutative}) \\ &= t_1 \lim_{v \rightarrow \infty} t^{(v)}(x) \\ &= t_1(z) . \end{aligned}$$

Thus $S(z) \cap T(z)$ consists of more than one point. This yields a contradiction to the fact that E is a product by induction.

Therefore $S(x) \cap T(x) = \{x\} \forall x \in \Omega$ and there is a natural biholomorphic map $\varphi : S(x_0) \times T(x_0) \rightarrow \Omega$ by

$$(s(x_0), t(x_0)) \mapsto (s \cdot t)(x_0) \quad (x_0 \in \Omega \text{ arbitrary}).$$

This yields a natural biholomorphic mapping

$$\tilde{Q} \times T \rightarrow \Omega$$

which is also denoted by φ . $\square$

By Step 4 it is clear that the closure of an S-orbit $S(x_0) \subset \Omega$ is a product Q of compact quadrics (Theorem 2.18) and that the closure $Y := \overline{T(x_0)}$ of a T-orbit $T(x_0) \subset \Omega$ is smooth.

Therefore we have an equivariant embedding

$$\tilde{Q} \times T \hookrightarrow Q \times Y.$$

Since all manifolds considered here as well as the group action are algebraic the biholomorphic mapping

$$\varphi : \tilde{Q} \times T \rightarrow \Omega$$

introduced in Step 5 can be extended to a bimeromorphic mapping

$$\bar{\varphi} : Q \times Y \dashrightarrow X.$$

We will prove that this extension is indeed biholomorphic.

Remark:

Let E be a G^0 -orbit of codimension one where the T -orbits are maximal-dimensional. Then for each $z \in E$ there exists exactly one S_1 -orbit $S_1(x) \subset \Omega$ having z in its closure by Lemma 2.8. Now if $\text{Iso}_2(T)$ is not trivial, for $e \neq t \in \text{Iso}_2(T)$ the S_1 -orbits $S_1(x)$ and $S_1(tx)$ have z in their closure. Contradiction.

Thus $\text{Iso}_2(T)$ is trivial.

Since the isotropies of an effective T -action on a torus embedding are connected ($[T-E]$), it follows from the product structure of the lower-dimensional G^0 -orbits that $\text{Iso}_X(T)$ is connected for all $x \in X$.

Step 6:

Let $A := \{x \in X \mid T(x) \cong (\mathbb{C}^*)^k\} \supset \Omega$.

Then there exists an equivariant holomorphic extension

$$\bar{\varphi} : \overline{S(x_0)} \times T(x_0) \rightarrow A$$

of the mapping φ introduced in Step 5.

Proof:

There is a natural biholomorphic mapping

$$\pi : \overline{S(x_0)} \times T(x_0) \rightarrow T \times \overline{S(x_0)}.$$

Since T is a complex Lie group acting holomorphically the mapping

$$\Psi : T \times \overline{S(x_0)} \rightarrow X, (t, x) \mapsto t(x)$$

is holomorphic.

Since for each point $y \in \overline{S(x_0)}$ the T-orbits are maximal-dimensional we have

$$\Psi(T \times \overline{S(x_0)}) \subset A \quad (\text{cf. Step 4}).$$

On the other hand, let $y \in A \setminus \Omega$. Then y is contained in the closure of an S-orbit $S(\tilde{y})$ in a G^0 -orbit of codimension one by induction.

This orbit is contained in the closure of an S-orbit $S(x) \subset \Omega$ by Step 1. Thus each $y \in A$ is contained in the closure of an S-orbit $S(x)$ with $x \in \Omega$. Thus $\Psi(T \times \overline{S(x_0)}) = A$.

Now Ψ is injective since the T-isotropies in A are trivial by the above Remark.

This proves that Ψ is biholomorphic.

Now since $\bar{\varphi}$ and $\Psi \circ \pi$ coincide on the open orbit the equivariance of both mappings implies that

$$\bar{\varphi}|(\overline{S(x_0)} \times T(x_0)) : \overline{S(x_0)} \times T(x_0) \rightarrow A$$

is indeed biholomorphic. $\square$

Define

$$B := \{y \in X \mid S(y) \cong \tilde{Q}\} \supset \Omega$$

We claim that

$$B = \{y \in \overline{T(x)} \mid x \in \Omega\}.$$

If $y \in B \setminus \Omega$ then there exists a G^0 -orbit E of codimension one and $z \in E$ such that $y \in \overline{T(z)}$ by induction. Of course the S-orbits in E are isomorphic to $\tilde{Q}$. Thus by faithful linearization there exists a T-orbit $T(x) \subset \Omega$ with $z \in \overline{T(x)}$. Therefore $y \in \overline{T(x)}$ and we have

$$B \subset \{y \in \overline{T(x)} \mid x \in \Omega\}.$$

For the other direction note that by Step 2 we find a G^0 -orbit E of codimension one with $\overline{T(x)} \cap E \neq \emptyset$. So each $z \in \overline{T(x)} \cap E$ is contained in B since $\overline{T(x)} \setminus T(x)$ consists of lower-dimensional T -orbits. If we assume that $\overline{T(x)} \setminus T(x)$ is not connected we have $T(x) \cong \mathbb{C}^*$, and $\overline{T(x)} \setminus T(x)$ consists of two points one of them contained in B . But since the symmetry s_x exchanges the two connected components of $\overline{T(x)} \setminus T(x)$, the other point has also to be contained in B . Thus $\overline{T(x)} \subset B$.

If $\overline{T(x)} \setminus T(x)$ is connected the set $\overline{T(z)}$ is positive-dimensional. The product structure of $\bar{E}$ yields that $\overline{T(z)} \subset B$. By faithful linearization we find a T -orbit $T(z')$ of codimension one in $\overline{T(x)}$ such that $\overline{T(z')} \cap \overline{T(z)} \neq \emptyset$. Since the S -isotropies in $\overline{T(z)}$ are of minimal dimension the same is true for the S -isotropies in $T(z')$. Since $T(z')$ is contained in a G^0 -orbit of codimension at least one, by induction we have $\overline{T(z')} \subset B$. Now the connectedness of $\overline{T(x)} \setminus T(x)$ yields that this process can be repeated until we have shown $\overline{T(\tilde{z})} \subset B$ for each T -orbit $T(\tilde{z})$ of codimension one in $\overline{T(x)}$. Thus $\overline{T(x)} \subset B$ since $T(x)$ is Stein.

Step 7:

Consider the bimeromorphic mapping

$$\bar{\varphi} : S(x_0) \times \overline{T(x_0)} \hookrightarrow X.$$

Then $\bar{\varphi} : S(x_0) \times \overline{T(x_0)} \rightarrow B$ is biholomorphic.

Proof:

As in the proof of Step 6 it follows that the mapping

$$\hat{\varphi} : S \times \overline{T(x_0)} \rightarrow X, (s, x) \mapsto s(x)$$

is holomorphic.

Since $B = \{y \in \overline{T(x)} \mid x \in \Omega\}$ it follows that

$$\hat{\varphi}(S \times \overline{T(x_0)}) = B.$$

Now the mapping

$$p : S \times \overline{T(x_0)} \rightarrow S(x_0) \times \overline{T(x_0)}$$

is holomorphic, surjective, and $\hat{\varphi}$ factors through p , since $\overline{T(x_0)}$ is a connected component of the fixed point set of $\text{Iso}_{x_0}(S)$, i.e. we have a diagram

$$\begin{array}{ccc} S \times \overline{T(x_0)} & \xrightarrow{p} & S(x_0) \times \overline{T(x_0)} \\ & \searrow \hat{\varphi} & \downarrow \psi \\ & & B \end{array}$$

It follows that ψ is continuous and bijective and that

$$\psi|_{S(x_0) \times \overline{T(x_0)}} = \hat{\varphi}|_{S(x_0) \times \overline{T(x_0)}}$$

Therefore $\hat{\varphi} : S(x_0) \times \overline{T(x_0)} \rightarrow B$ is indeed biholomorphic. $\square$

Remark:

Let $Q \times Y := \overline{S(x_0)} \times \overline{T(x_0)}$, $\tilde{Q} := S(x_0)$ and

$$\begin{aligned} F &:= \{x \in Q \times Y \mid x \notin (\tilde{Q} \times Y) \cup (Q \times T(x_0))\} \\ &= \{x \in Q \times Y \mid \text{Iso}_x(T) \text{ is not trivial and } \text{Iso}_x(S) \text{ is} \\ &\quad \text{not reductive}\}. \end{aligned}$$

Then we have shown that the restriction of the bimeromorphic mapping

$$\hat{\varphi} : Q \times Y \hookrightarrow X$$

to $(Q \times Y) \setminus F$ is indeed biholomorphic onto $A \cup B$.

Consider the projections

$$\pi_1 : Q \times Y \rightarrow Q, \quad \pi_2 : Q \times Y \rightarrow Y.$$

Then there are meromorphic mappings

$$\varphi_T := \pi_1 \circ \bar{\varphi}^{-1}$$

$$\varphi_S := \pi_2 \circ \bar{\varphi}^{-1}$$

which are nothing but the meromorphic extensions of the geometric quotient by the T-orbits (resp. S-orbits) introduced above

We will show that φ_T (resp. φ_S) are holomorphic.

For this we need:

Step 8:

Each $y \in X \setminus (A \cup B)$ is contained as well in $(\overline{A \setminus \Omega})$ as in $(\overline{B \setminus \Omega})$.

Proof:

It is enough to prove the statement for y in a G^0 -orbit E of codimension two in $X \setminus (A \cup B)$ since this set is dense in $X \setminus (A \cup B)$.

Then $\text{Iso}_y(T)$ is a one-parameter subgroup T_1 of T and $\text{Iso}_y(S)$ contains exactly one parabolic factor P_1 of S .

By looking at the induced action of T_1 and of a maximal torus D_1 of P_1 we find a decomposition

$$T_y(X) = T_y(E) \oplus V_1 \oplus V_2 \quad (\text{with } \dim_{\mathbb{C}} V_i = 1)$$

invariant under the action of D_1 and T_1 .

Now if $y \in (\overline{A \setminus \Omega})$, then there exists a G^0 -orbit $\hat{E}_1$ of codimension one and a T-orbit $T(z) \subset \hat{E}_1$ such that $y \in \overline{T(z)}$. It follows that $\text{Iso}_z(S)$ contains exactly one parabolic factor. This must be P_1 .

Thus we can choose V_1 such that the T_1 -action on V_1 is not trivial but the D_1 -action is trivial.

Now consider the one-dimensional subset L which is defined in a neighbourhood $U(z)$ by faithful linearization of the V_2 -part. If we assume that the D_1 -action on V_2 is trivial the dimension of the connected component of the fixed point set of D_1 containing y and z is $n - \dim_{\mathbb{C}} G_1/P_1$ (n the complex dimension of X). But by Lemma 2.8 this set has dimension $n - \dim_{\mathbb{C}} G_1/H_1$. Thus the D_1 -action on V_2 is not trivial.

Now for $x \in L \cap \Omega$ the $D_1 T_1$ -orbit through x is two-dimensional and thus $L \cap \Omega = \emptyset$ since L is stable under a maximal compact subgroup of $D_1 T_1$. Therefore $L \setminus \{y\}$ is contained in a G^0 -orbit $\hat{E}_2$ of codimension one. It is easy to see that T_1 acts trivially on $\hat{E}_2$ and that the S -orbits in E_2 are affine quadrics. Thus $y \in (\overline{B \setminus \Omega})$.

Now if $y \in (\overline{B \setminus \Omega})$, we again find a G^0 -orbit $\hat{E}_1$ of codimension one and $z \in \hat{E}_1$ such that $y \in \overline{S_1(z)}$. Consequently we can choose V_1 such that the T_1 -action on V_1 is trivial and the D_1 -action is not.

Now the T_1 -action on V_2 cannot be trivial. Otherwise T_1 acts trivially on X contradicting the effectivity of the T -action. Let L be the one-dimensional subspace corresponding to V_2 as above. Again $L \cap \Omega = \emptyset$ and $L \setminus \{y\}$ is contained in a G^0 -orbit $\hat{E}_2$ of codimension one. It is easy to see that the T -orbits in $\hat{E}_2$ are maximal-dimensional and that $y \in \overline{T_1(\tilde{z})}$ for a suitable $\tilde{z} \in \hat{E}_2$. Thus $y \in (\overline{A \setminus \Omega})$. $\square$

Consider the mapping

$$\tau : \tilde{Q} = Q_{(m_1)} \times \dots \times Q_{(m_n)} \rightarrow \prod_{i=1}^n \text{Gr}(m_i)$$

introduced in Lemma 2.16 and let Γ be the graph of this holomorphic mapping and $\Gamma' := \overline{\tau(\tilde{Q})} \subset \prod_{i=1}^n \text{Gr}(m_i)$.

We have shown that τ extends holomorphically to Q and that the graph of this holomorphic mapping is the irreducible component of

$$\Gamma := \{(x, h) \in Q \times \prod_{i=1}^n \text{Gr}(m_i) \mid h \subset \text{iso}_x(S)\}$$

containing Γ .

Moreover we claim (cf. Lemma 2.17)

Lemma 3.4:

For each $x \in Q$ there exists a neighbourhood $U(x) \subset Q$ such that

$$h \subset \text{iso}_x(S) \Rightarrow h \not\subset \text{iso}_y(S) \quad \forall y \in U(x), y \neq x.$$

Proof:

We have $h = (h_1, \dots, h_n)$ and $\text{iso}_x(S) = \text{iso}_x(S_1) \oplus \dots \oplus \text{iso}_x(S_n)$ and $x = (x_1, \dots, x_n)$. Thus $h \subset \text{iso}_x(S) \Rightarrow h_i \subset \text{iso}_{x_i}(S_i)$, $i = 1, \dots, n$.

Let $S_i(x_i)$ be non-compact. Then $h_i = \text{iso}_{x_i}(S_i)$ and the finiteness of $N_{S_i}(H_i)/H_i$ ($H_i := \text{Iso}_{x_i}(S_i)$) implies that there exists a neighbourhood $U(x_i) \subset Q_{m_i}$ with $h_i \not\subset \text{iso}_{y_i}(S_i) \quad \forall y_i \in U(x_i) \setminus \{x_i\}$. Now let $S_i(x_i)$ be compact and assume that there exists y_i with

$$h_i \subset \text{iso}_{x_i}(S_i) \cap \text{iso}_{y_i}(S_i).$$

If y_i is contained in the compact S_i -orbit, then

$$h_i = \text{Iso}_{x_i}(S_i) \cap \text{Iso}_{y_i}(S_i)$$

and the corresponding closed Lie group has a finite number of fixed points in the compact orbit by Lemma 2.15.

If y_i is contained in the open S_i -orbit then $h_i = \text{Iso}_{y_i}(S_i)$ and the corresponding group has a finite number of fixed points in the open orbit.

Thus there exists a neighbourhood $U(x_i) \subset Q_{m_i}$ with

$$h_i \notin \text{Iso}_{y_i}(S_i) \quad \forall y_i \in U(x_i) \setminus \{x_i\}.$$

Lemma 3.4 follows by defining

$$U(x) := \bigcup_{i=1}^n U(x_i). \quad \square$$

Lemma 3.5:

The mapping $\varphi_T : X \rightarrow Q$ is holomorphic.

Proof:

Since we already know that $\varphi_T|_A$ is holomorphic we consider $x \in X \setminus A$.

By Step 8 each $x \in X \setminus A$ is contained in the closure of an S -orbit $S(y) \subset A$. Then there exists a unique extension of $\varphi_T|_{S(y)}$ by Lemma 2.14.

This defines a point $\tilde{\varphi}_T(x)$ contained in $\varphi_T(x)$ with the property

$$\text{Iso}_x(S) = \text{Iso}_{\tilde{\varphi}_T(x)}(S).$$

We will prove that the irreducible analytic set $\varphi_T(x)$ contains $\tilde{\varphi}_T(x)$ as an isolated point.

Let $\sigma : \Omega \rightarrow \bigcup_{i=1}^n \text{Gr}(m_i)$ be the mapping

$$x \mapsto \text{Iso}_x(S).$$

This mapping is perhaps only meromorphically extendable.

Now consider sequences $(x_\nu)_{\nu \in \mathbb{N}} \subset A$ with

$$\lim_{\nu \rightarrow \infty} x_\nu = x .$$

Then each cluster point h of $(\sigma(x_\nu))_{\nu \in \mathbb{N}}$ satisfies

$$h \subset \text{iso}_x(S) .$$

Now for a cluster point z of $(\varphi_T(x_\nu))_{\nu \in \mathbb{N}}$ we also have

$$h \subset \text{iso}_z(S) .$$

But by Lemma 3.4 there exists a neighbourhood

$$U(\tilde{\varphi}_T(x)) \subset Q$$

such that

$$h \subset \text{iso}_{\tilde{\varphi}_T(x)}(S) = \text{iso}_x(S) \Rightarrow h \not\subset \text{iso}_z(S) \quad \forall z \in U(\tilde{\varphi}_T(x)) , z \neq \tilde{\varphi}_T(x) .$$

Thus $\tilde{\varphi}_T(x)$ is isolated in the irreducible analytic set $\varphi_T(x)$.

Thus φ_T is holomorphic. $\square$

Now we consider the meromorphic mapping

$$\varphi_S : X \rightarrow Y .$$

Since $\varphi_S|_{A \cup B}$ is holomorphic by the extension Theorem of Thullen-Remmert-Stein ([Siu]) divisors in Y can be lifted to S -invariant divisors in X . This is done by lifting the divisors at the points where the mapping is holomorphic. The closure of this set is analytic by the extension Theorem mentioned above and thus a divisor.

In order to show that φ_S is holomorphic we will show that for each point $z \in X \setminus (A \cup B)$ there exists a point $z' \in \varphi_S(z)$ with the property:

(*) If a divisor D does not contain z' then the lifted divisor does not contain z .

Let $z' \neq z'' \in \varphi_S(z)$. Then there exists a divisor D with $z' \notin D$, $z'' \in D$ since by our assumptions $Y \hookrightarrow \mathbb{P}_N$. But the lifted divisor $\tilde{D}$ does not contain z by (*). Contradiction. Therefore $\varphi_S(z)$ is one point and φ_S is holomorphic.

So we have to prove that for each z there exists $z' \in \varphi_S(z)$ with property (*).

By Steps 6 and 7 we already know that

$\varphi_S|_{(A \cup B)}$ is holomorphic.

So we only have to consider $z \in X \setminus (A \cup B)$.

Then $\tilde{T} := \text{Iso}_Z(T)$ is not trivial and $\text{Iso}_Z(S)$ is not reductive.

First we show that there is a one-parameter subgroup $\tilde{T}_z \subset \tilde{T}$ such that $z \in \overline{\tilde{T}_z(x)}$ for a suitable $x \in A$.

For this, note that by Step 8 there exists $\tilde{x} \in A$ such that $z \in \overline{T(\tilde{x})}$. Since $\text{Iso}_Z(S)$ is not reductive the set $\overline{T(\tilde{x})}$ is contained in the closure of a G^0 -orbit of codimension one.

It follows from the smoothness of $\overline{T(\tilde{x})}$ (induction) that there exist exactly $\ell := \dim_{\mathbb{C}} \tilde{T}$ T -stable divisors in $\overline{T(\tilde{x})}$ containing z . In suitable coordinates for T the one-parameter subgroups defining these divisors are

$$T_1 := \{(\lambda_1, 1, \dots, 1)\}, \dots, T_\ell := \{(1, \dots, 1, \lambda_\ell, 1, \dots, 1)\} \quad (\lambda_i \in \mathbb{C}^*)$$

If we take the algebraic one-parameter subgroup

$$\tilde{T}_z := \{(z_1, \dots, z_k) \in (\mathbb{C}^*)^k \mid z_1 = \dots = z_\ell; z_{\ell+1} = \dots = z_k = 1\},$$

by faithful linearization there exists $x \in A$ with $z \in \overline{\tilde{T}_z}(x)$.

Again by faithful linearization we find a small disc Δ such that

$$z \in \Delta \text{ and } \Delta \cap \tilde{T}_z(x) = \Delta \setminus \{z\} \subset A.$$

Since $\text{Iso}_z(S)$ is not reductive $\tilde{T}_z(x)$ lies in a G^0 -orbit E of codimension at least one which is contained in A . Then $\varphi_S|_E$ is holomorphic. The closure $\bar{E}$ is a product by induction and $\varphi_S|_E$ is the projection on the T -factor.

Thus $\varphi_S|_E$ can only be extended to $\bar{E}$ as the projection on the T -factor and $\varphi_S|(\Delta \setminus \{z\})$ can be extended to

$$\varphi_S : \Delta \rightarrow \tilde{\Delta} \quad (\tilde{\Delta} \text{ a small disc}).$$

Again by Step 8 the point z is contained in the closure of an S -orbit $S(y)$ with $\text{Iso}_z(T) = \text{Iso}_y(T)$ and $\text{Iso}_y(S)$ reductive. Thus φ_S is holomorphic near y since $y \in A \cup B$. Since $\tilde{T}_z \subset \text{Iso}_y(T)$, by faithful linearization there exists a unique $\tilde{T}_z$ -orbit in Ω containing y in its closure.

Now the disc $\tilde{\Delta}$ can be lifted uniquely to a disc $\Delta(y)$ with the following properties:

- (1) y is the center of $\Delta(y)$
- (2) $\varphi_S(\Delta(y)) = \tilde{\Delta}$
- (3) The $\tilde{T}_z$ -orbit in Ω , which contains y in its closure, contains $\Delta(y) \setminus \{y\}$.

In order to show this let $z' \in \Delta \setminus \{z\}$. Then there exists $y' \in \Omega$ such that $z' \in \overline{S(y')}$. Since $\varphi_S|_{A \cup B}$ is holomorphic the disc $\tilde{\Delta}$ can be lifted to a unique disc $\hat{\Delta}$ containing y' such that each point except one can be written as $t(y')$ ($t \in \mathbb{T}_z$).

Now let $s_\nu(y') \rightarrow z'$ and let $z'' := t(z')$ be another point in $\Delta \setminus \{z\}$, $y'' := t(y')$. Then we have $s_\nu(y'') \rightarrow z''$.

Therefore there is a sequence

$$\Delta_\nu := s_\nu(\hat{\Delta})$$

of discs which converges pointwise to Δ in each point except z . Thus $s_\nu(\hat{\Delta}) \rightarrow \Delta$ and we can choose y to be the center of $\hat{\Delta}$. Thus the existence of $\Delta(y)$ is proved.

Now let $\tilde{D}$ be a divisor in Y which does not contain $\varphi_S(y)$ (i.e. $\tilde{D}$ does not contain the center of the disc $\tilde{\Delta}$). After replacing $\tilde{\Delta}$ by a smaller disc, if necessary, we may assume that $\tilde{D} \cap \tilde{\Delta} = \emptyset$.

Now $\tilde{D}$ can be lifted to an S -invariant divisor D in X . By construction $D \cap \Delta(y) = \emptyset$.

The S -invariance of D yields that for each $y_\nu \in S(y)$ it follows that

$$D \cap \Delta(y_\nu) = \emptyset.$$

Here $\Delta(y_v)$ is the disc which has properties analogous to the properties of $\Delta(y)$ stated above. By our construction it follows that

$$y_v = s(y) \rightarrow s(\Delta(y)) = \Delta(y_v) .$$

Now choose a sufficiently small neighbourhood $U(z)$ such that

$$D \cap U(z) = \{\tilde{z} \in U(z) \mid f(\tilde{z}) = 0\} .$$

If D is not irreducible, we consider the irreducible components separately.

By replacing the disc Δ by a smaller disc if necessary, we may assume that $\Delta \subset U(z)$ and $\Delta(y_v) \subset U(z)$ for $y_v \in S(y)$ sufficiently near to z .

Defining

$$g_v := f|_{\Delta(y_v)}$$

we obtain a sequence of holomorphic functions.

The functions g_v can be regarded as functions on the unit disc, since $\varphi_S(\Delta(g_v)) = \tilde{\Delta}$.

From our construction it is clear that

$$g_v \rightarrow g := f|_{\Delta} \quad (\text{pointwise}) .$$

We may assume that f is bounded on $U(z)$. So the sequence g_v is a locally bounded sequence of holomorphic functions. Thus $(g_v)_{v \in \mathbb{N}}$ is a normal family of holomorphic functions. Then, by a Theorem of Vitali

$g_v \rightarrow g$ (uniform convergence on compact sets).

Now we assume that the divisor D contains z , i.e. $f(z) = 0$.

But g_v has no zeros by construction and the convergence

$$g_v \rightarrow g$$

is uniform.

Thus either g has no zeros or g is identically zero. Since $g(z) = 0$ by our assumption we have $g \equiv 0$, i.e. $f \equiv 0$ on Δ .

But then the divisor $\tilde{D}$ contains $\tilde{\Delta}$. Contradiction.

So we have found a divisor $\tilde{D}$ with $z \notin \tilde{D}$. Since $z \in X \setminus (A \cup B)$ was chosen arbitrarily, this proves

Lemma 3.6:

The meromorphic mapping

$$\varphi_S : X \rightarrow Y$$

is holomorphic. $\square$

Lemma 3.7:

The mapping

$$\psi := (\varphi_S, \varphi_T) : X \rightarrow Y \times Q$$

is biholomorphic.

Proof:

By Lemmas 3.5 and 3.6 it follows that ψ is holomorphic and surjective.

If $\varphi_S(x') = \varphi_S(x)$, then x' and x are both contained in a set of the form $\overline{S(y_1)}$ ($y_1 \in \Omega$).

If $\varphi_T(x') = \varphi_T(x)$, then x' and x are both contained in a set of the form $\overline{T(y_2)}$ ($y_2 \in \Omega$).

So $\psi(x) = \psi(x') \ (x \neq x')$ implies that

$$\overline{S(y_1)} \cap \overline{T(y_2)}$$

consists of more than one point.

This yields a contradiction to the fact that $S(y) \cap T(y) = \{y\}$ and that the closures of the S-orbits in Ω are contained in A and the closures of the T-orbits in B . Thus ψ is injective. $\square$

The above Lemma completes the proof of Theorem 3.1. $\square$

Part II:

C O M P L E X - S Y M M E T R I C
T O R U S
E M B E D D I N G S

§ 1. General Properties of Complex-symmetric Torus Embeddings.

1.1. Affine Torus Embeddings

An algebraic torus over $\mathbb{C}$ is an algebraic group T isomorphic to $(\mathbb{C}^*)^n$.

The algebra $\Gamma(\mathcal{O}_T)$ of regular functions on T is isomorphic to

$$B := \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}] .$$

Let

$M := \text{Hom}(T, \mathbb{C}^*)$ be the group of characters of T and

$N := \text{Hom}(\mathbb{C}^*, T)$ be the group of algebraic 1-parameter subgroups.

For $r \in M$ the corresponding regular function is denoted by x^r , and for $a \in N$ the corresponding homomorphism by $\lambda_a : \mathbb{C}^* \rightarrow T$.

The groups N and M are free abelian groups of rank n , which are dual via the non-singular pairing

$$[\cdot, \cdot] : M \times N \rightarrow \mathbb{Z}, \text{ defined by } x^r(\lambda_a(t)) = : t^{[r, a]} \quad (t \in \mathbb{C}^*) .$$

It is clear that the algebra B can also be written as

$$B = \mathbb{C}[\dots, x^r, \dots]_{r \in M} = : \mathbb{C}[M] .$$

So the T -action on T defines a gradation on B . If we have any affine variety X with a T -action, then the T -action on its function algebra A also defines a gradation, since the group is reductive and abelian and therefore each algebraic T -representation is the direct sum of one-dimensional representations (c.g. [Bo 1]).

Moreover, if $T \subset X$ as an open subset, we can assume $B \supset A$ and that the quotient field of A is equal to the quotient field of B ([T-E], p. 3).

Conversely, if we have a finitely generated subsemigroup S of characters in M then $\mathbb{C}[S]$ is the algebra of regular functions of an affine variety $X = \text{Spec } \mathbb{C}[S]$ with a torus-action.

In fact we have

Proposition 1 ([T-E], p. 4):

The correspondence $S \mapsto \text{Spec } \mathbb{C}[S]$ defines a bijection between

{finitely generated subsemigroups S of M which generate M as a group}

and

{isomorphism classes of affine torus embeddings of T } . $\square$

For example if we take the subsemigroup of $\mathbb{Z}$ generated by 2 and 3, then the corresponding variety is Neil's parabola $\text{Spec } \mathbb{C}[t^2, t^3]$.

This is a non-normal $\mathbb{C}^*$ -embedding.

A subsemigroup S of a lattice G is called saturated if

$nr \in S$ ($n \geq 1$, $r \in G$) implies $r \in S$.

Proposition 2 ([T-E], p. 5):

An affine torus embedding $X = \text{Spec } \mathbb{C}[S]$ is normal, if and only if S is saturated. $\square$

In the sequel we always assume that X is normal.

Saturated semigroups can be described using special cones

in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ or $M_{\mathbb{R}}$, respectively.

Definition:

A subset $\sigma \subset N_{\mathbb{R}}$ (resp. $M_{\mathbb{R}}$) is called a convex rational polyhedral cone or simply a cone, if one of the following equivalent conditions are fulfilled:

(1) There exist linear functionals $\ell_1, \dots, \ell_r$ defined over $\mathbb{Q}$ such that

$$\sigma = \{x \mid \ell_i(x) \geq 0, i = 1, \dots, r\}.$$

(2) There exist vectors $v_1, \dots, v_s$ defined over $\mathbb{Q}$ such that

$$\sigma = \{x \mid x = \sum_{i=1}^s \lambda_i v_i, \lambda_i \geq 0, i = 1, \dots, s\} =: \langle v_1, \dots, v_s \rangle.$$

Examples:

(1) The cone $\langle (1, -1), (-1, 3) \rangle \subset \mathbb{R}^2$ is defined by the linear functionals

$$\ell_1(x_1, x_2) = x_1 + x_2, \ell_2(x_1, x_2) = 3x_1 + x_2$$

(2) The upper halfplane $\sigma = \{(x_1, x_2) \mid \ell_1(x) := x_2 \geq 0\} \subset \mathbb{R}^2$ is the smallest cone containing the vectors $(1, 0), (-1, 0), (0, 1)$,

$$\sigma = \langle (1, 0), (-1, 0), (0, 1) \rangle.$$

(3) The right halfline $\sigma = \{(x_1, 0) \mid x_1 \geq 0\}$ can be written as

$$\sigma = \langle (1, 0) \rangle \text{ or equivalently}$$

$$\sigma = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, -x_2 \geq 0\}.$$

Definition:

A cone σ that can be written as

$\sigma = \langle v_1, \dots, v_s \rangle$ with linear independent vectors $v_1, \dots, v_s$ is called a simplicial cone.

Examples (1) and (3) are simplicial while Example (2) is not.

If σ is a cone in $N_{\mathbb{R}}$ then we can define the dual cone $\check{\sigma} \subset M_{\mathbb{R}}$ by

$$\check{\sigma} := \{r \in M_{\mathbb{R}} \mid [r, a] \geq 0 \ \forall a \in \sigma\}$$

(There is a natural extension of the pairing $[,] : M \times N \rightarrow \mathbb{Z}$ to a pairing $[,] : M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$).

If σ is defined by the linear functionals $\ell_1, \dots, \ell_r$ then the dual cone $\check{\sigma}$ can be described by

$$\check{\sigma} = \left\{ \sum_{i=1}^r \lambda_i \ell_i \mid \lambda_i \geq 0, i = 1, \dots, r \right\}.$$

Examples:

(1) $\check{\sigma} := \langle (1,1), (3,1) \rangle$ is the dual cone to the cone in Example (1) above.

(2) The dual cone to the upper half-plane is $\check{\sigma} = \langle (0,1) \rangle$.

(3) The dual cone to the right half-line is the right halfplane.

The following properties of dual cones are immediate:

(1) $\check{\sigma}$ does not contain any linear subspace except $\{0\} \Leftrightarrow \sigma$ is not contained in a proper linear subspace of $N_{\mathbb{R}}$.

(2) $\check{\check{\sigma}} = \sigma$

It can be shown that if $\sigma \subset N_{\mathbb{R}}$ is a cone which does not contain a linear subspace then $\check{\sigma} \cap M$ is a finitely generated (by Gordon's Lemma ([Dan]) saturated subsemigroup which generates M as a group (see the first property of dual cones above) and vice versa.

Combining this with Proposition 1 we obtain

Proposition 3 ([T-E], p. 8):

The correspondence $\sigma \rightarrow \text{Spec } \mathbb{C}[\check{\sigma} \cap M] = : X_{\sigma}^{\vee}$ defines a bijection between

{rational convex polyhedral cones in $N_{\mathbb{R}}$ which do not contain any non-trivial linear subspaces}

and

{normal affine torus embeddings} . $\square$

Examples:

(1) If $\sigma = \langle (1, -1), (-1, 3) \rangle$ then

$$X_{\sigma} = \text{Spec } \mathbb{C}[x_1, x_2, x_1^2 x_2, x_1^3 x_2] ,$$

since $\sigma \cap M$ is generated by the elements $(1, 1), (2, 1)$ and $(3, 1)$.

This variety can be identified with

$$X := \{(x, y, z) \mid xz = y^2\} \subset \mathbb{C}^3$$

and the T-action on X is defined by

$$(\lambda, \mu)(x, y, z) := (\lambda \mu x, \lambda^2 \mu y, \lambda^3 \mu z) .$$

The variety X is not smooth since the fixed point $(0, 0, 0)$ of T is a singular point of X .

(2) If $\sigma = \langle (1, 0), (0, 1) \rangle$, then X_{σ} is isomorphic to $\mathbb{C}^2$ with the usual embedding of $(\mathbb{C}^*)^2$.

(3) If $\sigma = \{(0, \dots, 0)\} = \langle 0 \rangle$, then X_{σ} is isomorphic to T with the natural T-action.

The following Proposition shows that the smoothness of an affine torus embedding X_{σ} can be read off from the defining cone σ .

Proposition 4 ([T-E], p. 14):

The variety X_{σ} is smooth, if and only if $\sigma \cap N$ can be generated by a subset of a basis of N over $\mathbb{Z}$. $\square$

Remark:

Since the cones are defined over $\mathbb{Q}$ the generating vectors of σ can be chosen as primitive elements of N (an element n of a lattice is called primitive, if $rn \in N$ ($r \in \mathbb{R}$) implies $r \in \mathbb{Z}$).

Examples.

(1) The cone $\sigma = \langle (1, -1), (-1, 3) \rangle$ defines a singular embedding since $(1, -1), (-1, 3)$ is not a $\mathbb{Z}$ -basis of $N \cong \mathbb{Z}^2$.

(2) The cone $\sigma = \langle (1, 0) \rangle$ defines a non-singular torus embedding (X_{σ} is isomorphic to $\mathbb{C}^* \times \mathbb{C}$).

Many important properties of torus embeddings can be stated in terms of the faces of a cone.

Definition:

If $\sigma = \{x \mid \ell_i(x) \geq 0, i = 1, \dots, r\}$ and if $\{i_1, \dots, i_k\} \subset \{1, \dots, r\}$, then

$$\tau := \{x \in \sigma \mid \ell_{i_1}(x) = \dots = \ell_{i_k}(x) = 0\}$$

is again a cone which is called a face of σ .

We write $\tau < \sigma$.

For example the cone $\langle (1,0), (0,1) \rangle$ has the faces $\tau_1 = \langle (1,0) \rangle$, $\tau_2 = \langle (0,1) \rangle$ and $\tau_3 = \langle (0,0) \rangle$.

It is known that equivariant mappings between tori induce homomorphisms of the corresponding groups of one-parameter subgroups and characters. This can be used to describe equivariant mappings between torus embeddings.

Proposition 5 ([Dan]; p. 105):

(1) Let X_σ (resp. $X_{\sigma'}$) be affine torus embeddings of T . Then there exists an equivariant map

$$f : X_\sigma \rightarrow X_{\sigma'}, \text{ (with } f(T) \cong T),$$

if and only if there exists an isomorphism (of groups) $\varphi : N \rightarrow N'$ such that its scalar extension satisfies $\varphi(\sigma) \subset \sigma'$.

(2) The mapping f is a bijection, if and only if $\varphi(\sigma) = \sigma'$.

(3) There exists a mapping $g : X_\sigma \rightarrow X_{\sigma'}$, such that the diagram

$$\begin{array}{ccc} & & X_\sigma \\ & \nearrow & \downarrow \\ T & & X_{\sigma'} \end{array} \quad \text{commutes,}$$

if and only if $\sigma \subset \sigma'$.

(4) The mapping g is an open immersion, if and only if $\sigma < \sigma'$. $\square$

Remarks:

(1) Descriptions of more general mappings between torus embeddings can for example be found in [Dan] (p. 104 f.).

(2) An automorphism $f : X_\sigma \rightarrow X_\sigma$ is induced by the translations in T , if and only if the induced mapping φ is the identity.

(3) As a consequence of the above Proposition the faces of σ correspond to open T -stable subvarieties immersed in X_σ (e.g. the face $\{0\}$ always corresponds to $T \subset X_\sigma$).

The following connection between the faces of σ and T-orbits in X_σ can be proved using 1-parameter subgroups, i.e. elements of N .

Proposition 6 ([Oda], p. 11):

Let σ be a cone with corresponding affine torus embedding X_σ . Then there exists a bijective mapping

$$\text{orb} : \{\text{faces of } \sigma\} \rightarrow \{\text{T-orbits in } X_\sigma\}$$

with the following properties:

$$\text{orb}(\langle 0 \rangle) = T$$

$$\tau_1 < \tau_2 \Rightarrow \text{orb}(\tau_2) \subset \overline{\text{orb}(\tau_1)}$$

$$\dim \tau + \dim \text{orb}(\tau) = \dim T. \quad \square$$

Example:

If σ is again the cone $\langle (1,0), (0,1) \rangle$, which defines $\mathbb{C}^2$, the one-dimensional cones τ_1 and τ_2 correspond to the one-codimensional $(\mathbb{C}^*)^2$ -orbits and the cone σ itself to the fixed point $0 \in \mathbb{C}^2$.

Combining Proposition 6 and Proposition 4 we obtain

Proposition 7:

If X_σ is smooth, the closure of each T-orbit in X_σ is again a smooth variety. $\square$

A similar property is valid for normal affine torus embeddings:

Proposition 8 ([T-E], p. 16):

If X_σ is a normal affine torus embedding, then the closure of each T-orbit in X_σ is again a normal variety.

Remark:

This is a special case of a more general Theorem ([B-P]).

1.2. General Torus Embeddings

We now consider general algebraic varieties which contain T as an open orbit and where the T -action on T by translations can be extended to an action on X . This action is effective by definition. Note that algebraic only means Moishezon, not projective algebraic.

An algebraic variety is obtained by patching together affine algebraic varieties. The fundamental fact that enables us to reduce the case of varieties to the affine case is the following special case of a Theorem due to Sumihiro.

Theorem 10 ([Oda], p. 3; [T-E], p. 20):

If an algebraic torus T acts algebraically on a normal algebraic variety X (locally of finite type), then X is covered by T -stable affine open subsets. $\square$

Remark:

The normality of X is essential ([Oda], p. 4).

The above result shows that a normal torus embedding can be described (at least locally) using cones. If U_i and U_j are T -stable affine subsets then $U_i \cap U_j$ is again a T -stable affine subset immersed in U_i as well as in U_j . So by Proposition 5 the following definition makes sense.

Definition:

A fan (or rational partial polyhedral decomposition) in $N_{\mathbb{R}}$ is a system $\Sigma = \{\sigma\}$ of cones satisfying the following conditions

- (1) each cone does not contain any linear subspace except $\{0\}$;
- (2) if $\sigma \in \Sigma$ and τ is a face of σ , then $\tau \in \Sigma$;
- (3) if $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2 \in \Sigma$ and $\sigma_1 \cap \sigma_2$ is a face of σ_1 and as well of σ_2 .

A fan is called simplicial, if it consists of simplicial cones.

Remark:

The first property ensures that all T -stable affine varieties contain T as an open orbit. The other properties allow us to glue together the $X_{\sigma} (\sigma \in \Sigma)$ to form a normal variety.

This variety is denoted by X_{Σ} .

Indeed it is possible to describe all normal torus embeddings using fans.

Proposition 11 ([Oda], p. 16, p. 10):

(1) There exists a bijection

$$\{\text{fans } \Sigma \text{ in } N_{\mathbb{R}}\} \rightarrow \{\text{normal varieties which are torus embeddings } X_{\Sigma} \text{ of } T \text{ (locally of finite type)}\}$$

(2) An equivariant mapping $f : X_{\Sigma} \rightarrow X_{\Sigma'}$, between normal torus embeddings of T (with $f|T$ bireg.) induces an isomorphism $\varphi : N \rightarrow N'$ such that its scalar extension $\varphi : N_{\mathbb{R}} \rightarrow N'_{\mathbb{R}}$ satisfies

$$\varphi(\Sigma) \subset \Sigma' : \Leftrightarrow \varphi(\sigma) \subset \sigma' \in \Sigma' \quad \forall \sigma \in \Sigma$$

and vice versa. $\square$

Remark:

(2) follows immediately from Proposition 5.

As a special case automorphisms $f : X_{\Sigma} \rightarrow X_{\Sigma}$ induce isomorphisms $\varphi : N \rightarrow N$ such that

$$\varphi(\Sigma) = \Sigma \Leftrightarrow \varphi(\sigma) = \sigma : \sigma' \in \Sigma \quad \forall \sigma \in \Sigma.$$

The automorphism f induces $\varphi = \text{Id}$, if and only if f is induced by a translation in T .

Since for each $\sigma \in \Sigma$ the faces of σ are also elements of Σ , the following Proposition is an immediate consequence of Propositions 5 and 6.

Proposition 12 ([T-E], p. 24):

(1) The map

$$\Sigma \ni \sigma \mapsto X_{\sigma} \subset X_{\Sigma}$$

defines a bijection between the cones of Σ and T -stable open affine subsets of X_{Σ} .

(2) There exists a map

$$\text{orb} : \Sigma \rightarrow \{T\text{-orbits in } X_{\Sigma}\}$$

with properties analogous to those in Proposition 6. $\square$

Since smoothness and normality are local properties, Propositions 4, 7 and 8 imply the following

Proposition 13:

- (1) For $\sigma \in \Sigma$ the closure $\overline{\text{orb}(\sigma)}$ of $\text{orb}(\sigma)$ is a normal variety.
- (2) The fan Σ defines a smooth torus embedding, if and only if for each $\sigma \in \Sigma$ the semigroup $\sigma \cap N$ can be generated by a subset of a basis of N over $\mathbb{Z}$.
- (3) If X_Σ is smooth, $\overline{\text{orb}(\sigma)}$ is smooth $\forall \sigma \in \Sigma$. $\square$

Remark:

The variety $\overline{\text{orb}(\sigma)}$ is not in general affine.

Before giving some examples, we state

Proposition 14 ([Dan]; p. 114):

The fan Σ defines a complete torus embedding, if and only if Σ is finite and $\text{supp}(\Sigma) := \bigcup_{\sigma \in \Sigma} \sigma = N_{\mathbb{R}}$.

In this case Σ is called a complete fan.

Examples:

- (1) Consider the complete fan Σ in $\mathbb{R}$ defined by

$$\Sigma := \{ \langle 0 \rangle, \langle 1 \rangle, \langle -1 \rangle \} = \{ \langle 0 \rangle, \sigma_1, \sigma_2 \}.$$

Then X_{σ_1} and X_{σ_2} are isomorphic to $\mathbb{C}$ and since $\sigma_1 \cap \sigma_2 = \{0\}$, their intersection is isomorphic to $\mathbb{C}^*$.

Moreover, if $X_{\sigma_1} \cong \text{Spec } \mathbb{C}[X]$, the change of coordinates is defined by $X \mapsto X^{-1}$. Therefore $X_\Sigma \cong \mathbb{P}_1$.

Equivariant automorphisms induce either $\varphi = \text{Id}$ or $\varphi = -\text{Id}$.

The latter automorphisms exchange the points 0 and ∞ of $\mathbb{P}_1$.

- (2) More generally, if $N = \mathbb{Z}^n$, $e_1, \dots, e_n$ is a $\mathbb{Z}$ -basis and $e_0 := -(e_1 + \dots + e_n)$, then

$$\Sigma := \{ \langle e_{i_1}, \dots, e_{i_k} \rangle \mid i_j \in \{0, \dots, n\}, 0 \leq k \leq n \}$$

defines a complete non-singular fan.

The associated variety is $X_\Sigma \cong \mathbb{P}_n$.

- (3) For $a \in \{0, 1, \dots\}$ consider the fan

$$\Sigma_a := \{ \langle 0 \rangle, \langle e_1 \rangle, \langle e_2 \rangle, \langle -e_2 \rangle, \langle -e_1 + ae_2 \rangle, \langle e_1, e_2 \rangle, \langle e_1, -e_2 \rangle, \langle -e_2, -e_1 + ae_2 \rangle, \langle -e_1 + ae_2, e_2 \rangle \}.$$

This fan defines a two-dimensional complete non-singular torus embedding which is isomorphic to the Hirzebruch surface F_a ($a \geq 0$).

Remark:

If a vector v which can be chosen to be a primitive element of N defines a simplicial cone $\sigma \in \Sigma$, then $\langle v \rangle \in \Sigma$. These one-dimensional cones are called rays. By Proposition 13 a ray defines a T -stable divisor in X_Σ .

The system of all rays is often called the skeleton of Σ .

$$\text{Sk}(\Sigma) = \{v | \langle v \rangle \in \Sigma, v \in N, v \text{ primitive}\}.$$

Products of torus embeddings can also be described using fans.

Proposition 15 ([Oda], p. 39):

Let T' (resp. T'') be algebraic tori, N' (resp. N'') the corresponding groups of one-parameter subgroups and Σ' (resp. Σ'') fans in $N'_{\mathbb{R}}$ (resp. $N''_{\mathbb{R}}$).

Then $X_{\Sigma'} \times X_{\Sigma''}$ is a normal torus embedding for $T' \times T''$ and the corresponding fan is obtained by

$$\Sigma := \Sigma' \times \Sigma'' := \{\sigma' \times \sigma'' | \sigma' \in \Sigma', \sigma'' \in \Sigma''\}.$$

Example:

It is easy to see that the fan Σ_0 (Example (3) above) can be written as

$$\Sigma_0 = \{\langle 0 \rangle, \langle e_1 \rangle, \langle -e_1 \rangle\} \times \{\langle 0 \rangle, \langle e_2 \rangle, \langle -e_2 \rangle\}.$$

Therefore $X_{\Sigma_0} = F_0 \cong \mathbb{P}_1 \times \mathbb{P}_1$.

The following description of the direct product is used in § 2.

Lemma 16:

Let Σ be a complete non-singular fan in $V := N_{\mathbb{R}}$.

Then Σ is the direct product

$$\Sigma = \Sigma' \times \Sigma''$$

of non-singular fans, if and only if there exist non trivial subspaces V', V'' of V with

$$(1) V = V' \oplus V'';$$

$$(2) \text{ each cone } \sigma \in \Sigma \text{ can be written as } \sigma = \pi_{V'}(\sigma) \times \pi_{V''}(\sigma)$$

(where π_W denotes the projection on the subspace W).

Moreover,

$$\text{Sk}(\Sigma) \cap V' = \text{Sk}(\Sigma') \quad \text{and} \quad \text{Sk}(\Sigma) \cap V'' = \text{Sk}(\Sigma'').$$

Proof:

If Σ is a product, then the existence of subspaces with these properties is obvious.

So we only have to prove the other direction.

Each cone $\sigma \in \Sigma$ can be written as

$$\sigma = \langle v_1', \dots, v_k', v_1'', \dots, v_\ell'' \rangle \text{ with } v_j' \in V', v_j'' \in V''.$$

Thus $\sigma \cap V' = \langle v_1', \dots, v_k' \rangle \in \Sigma$ is a non-singular cone in V' since $v_1', \dots, v_k'$ is part of a $\mathbb{Z}$ -basis of V' .

Therefore

$$\Sigma' := V' \cap \Sigma = \{\sigma \in \Sigma \mid \sigma \subset V'\} \text{ and } \Sigma'' := V'' \cap \Sigma$$

are well-defined complete non-singular fans in V' (resp. V'').

Of course each cone $\sigma \in \Sigma$ is contained in

$$\tilde{\Sigma} := \{\sigma' \times \sigma'' \mid \sigma' \in \Sigma', \sigma'' \in \Sigma''\}.$$

The Lemma is proved, if we can show $\Sigma = \tilde{\Sigma}$.

Now if we assume that there exists a maximal-dimensional cone

$\sigma' \times \sigma'' \in \tilde{\Sigma}$ which does not belong to Σ , then

$$\text{supp}(\Sigma) \subset ((V \setminus (\sigma' \times \sigma'')) \cup \{\text{faces of } \sigma' \times \sigma''\}) \subsetneq V.$$

Thus Σ is not complete, contrary to our assumption.

So all maximal-dimensional cones of $\tilde{\Sigma}$ are contained in Σ .

Since Σ and $\tilde{\Sigma}$ define fans, we have $\Sigma = \tilde{\Sigma}$. $\square$

Remarks:

(1) It is not sufficient that all cones are simplicial. At the end we will give a counterexample to Lemma 16 (cf. p. 155) in the case of normal singular embeddings.

(2) As the following example shows the completeness is necessary



We have

$$\text{Sk}(\Sigma) = \{e_1, e_2, -e_1, -e_2\},$$

but Σ is not a product.

The corresponding fan $\tilde{\Sigma}$

is the fan of $\mathbb{P}_1 \times \mathbb{P}_1$.

(3) Equivariant fiber-bundles of torus embeddings can be described in a similar fashion (we refer to [Oda], p. 59)

We now describe the σ -process of blowing up a fixed point of T in a two-dimensional smooth torus embedding.

Proposition 17 ([Oda], p.43):

Let X_Σ be a non-singular torus embedding of dimension two and $\sigma := \langle n_1, n_2 \rangle$ a cone defining a fixed point of the T -action. Let X_{Σ^*} be the torus embedding obtained by applying the σ -process of equivariantly blowing up this fixed point.

Then

$$\Sigma^* = (\Sigma \cup \{\langle n_1, n_1+n_2 \rangle, \langle n_1+n_2, n_2 \rangle, \langle n_1+n_2 \rangle\}) \setminus \{\langle n_1, n_2 \rangle\} . \quad \square$$

Example:

If Σ is the fan describing $\mathbb{P}_2$ take $\sigma = \langle -(e_1+e_2), e_1 \rangle$. Then $\text{Sk}(\Sigma^*) = \{e_1, e_2, -e_1-e_2, -e_2\}$ and the variety X_{Σ^*} is isomorphic to F_1 (Apply the linear transformation $\varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to Σ^* obtain the fan Σ_1).

Remark:

The generalized σ -process of blowing up a T -stable subvariety can be described in a similar way (cf. [Oda], p. 43).

For simplicity, the set of all cones in Σ which contain a fixed cone σ is called the star of σ ([Dan], p. 115)

$$\text{St}(\sigma) := \{\tau \in \Sigma \mid \tau \supset \sigma\} .$$

It follows from Proposition 13 that for $\sigma \in \Sigma$ the variety $\overline{\text{orb}}(\sigma)$ is again a normal variety and that $\overline{\text{orb}}(\sigma)$ is smooth, if Σ defines a smooth torus embedding.

In fact it is possible to obtain all relevant information about $\overline{\text{orb}}(\sigma)$ from the fan Σ .

Proposition 18 ([Dan], p. 115):

If $\sigma \in \Sigma$, then $\overline{\text{orb}}(\sigma)$ is again a torus embedding for the quotient torus T' which is obtained as the quotient of T by the ineffectivity of the T -action on $\text{orb}(\sigma)$.

The corresponding fan Σ_σ is obtained by projecting

$$\text{St}(\sigma) = \{\tau \in \Sigma \mid \tau \supset \sigma\}$$

on the factor-space

$N_{\mathbb{R}}/(\sigma)_{\mathbb{R}}$, where $(\sigma)_{\mathbb{R}}$ is the smallest linear subspace of $N_{\mathbb{R}}$ containing σ . $\square$

1.5. Complex-symmetric Torus Embeddings

From now on we want to consider normal compact algebraic torus embeddings X which are complex-symmetric, i.e. for each $x \in X$ there exists a holomorphic automorphism (which is called a symmetry) s_x with

(1) $s_x(x) = x$ and x is isolated fixed point

(2) $s_x^2 = \text{Id}$

(3) s_x respects the T -orbits.

It is clear that for any holomorphic G -action on a normal complex space X there is a representation

$\rho : G^0 \rightarrow \text{Aut}_0(X)$ such that

$\alpha(g(x)) = (\rho(g) \cdot \alpha)(x) \quad \forall x \in X, \forall g \in G^0, \forall \alpha \in \text{Aut}_0(X).$

Since $s_x(T) = T$ for each symmetry, it follows from Proposition 11 that each symmetry induces a fan automorphism.

These fan automorphisms will be described in detail later on.

The following Lemma is valid for arbitrary normal torus embeddings.

Lemma 19:

Let X be a normal torus embedding over $\mathbb{C}$. Then the isotropies of the T -action on X are connected.

Proof:

Denote the open T -orbit in X by Ω .

Then the Lemma is true for $x \in \Omega$ since the action of T on Ω is assumed to be effective.

Moreover, since the T -action is algebraic, all isotropies have finitely many connected components. Thus for each isotropy $\tilde{H}$ there exists a maximal compact subgroup K of $\tilde{H}$ such that K^0 is a maximal compact subgroup of $\tilde{H}^0$.

Now assume that the Lemma is proved for orbits of codimension at most k and take a T -orbit E_{k+1} of codimension $k+1$. Then there exists a T -orbit E_k of codimension k with $E_{k+1} \subset \overline{E_k}$. By Proposition 13 the variety $\overline{E_k}$ is normal, thus $E_{k+1} \cup E_k$ is smooth. Moreover the quotient torus $T' := T/T_1$ (with T_1 connected) acts effectively on $\overline{E_k}$.

Now for $x \in E_{k+1}$ define $H := \text{Iso}_x(T')$. It follows that the tangent space $T_x(E_k)$ splits into the sum $T_x(E_{k+1}) \oplus V$ of H -invariant subspaces where $\dim_{\mathbb{C}} V = 1$.

Since the action of H can be faithfully linearized ([H-O], p. 4), we obtain a faithful representation of H on $\mathbb{C}^*$. But this is only possible, if the group H is connected, since T' acts effectively on E_k (cf. part I, Lemma 2.6).

Thus H is connected.

Let $\pi : T \rightarrow T'$ be the canonical projection, then

$\tilde{T} := \pi^{-1}(H)$ is a connected subgroup of T defining the isotropy of the T -action on E_{k+1} . Therefore the Lemma is proved by induction. $\square$

From now on we will consider the complex-symmetric case.

First we have the following

Lemma 20:

Let X be a complete non-singular torus embedding. Then there always exists a symmetry for the fixed points of the T -action.

Proof:

Since the associated fan Σ is complete, there exist finitely many cones of maximal dimension in Σ . These correspond to the fixed points of T in X (by Proposition 12). Thus the set of fixed points of T is finite and not empty.

Let $p \in X$ be a fixed point of T and let $\sigma = \langle v_1, \dots, v_n \rangle$ be the cone of maximal dimension associated to p . Since X is smooth, the vectors $v_1, \dots, v_n$ form a $\mathbb{Z}$ -basis of N (Proposition 13) and, by a special choice of coordinates for T , we can assume that $v_i = (0, \dots, 0, 1, 0, \dots, 0)$.

The vectors v_i define T -orbits D_i of codimension one with $\bigcap_{i=1}^n \overline{D_i} = \{p\}$. It follows from Lemma 19 and from ([T-E], p. 10) that the ineffectivity H_i of the T -action on D_i is the one-parameter subgroup belonging to v_i (in these coordinates

$H_i = \{(1, \dots, 1, t_i, 1, \dots, 1) \mid t_i \in \mathbb{C}^*\}$). Moreover, if E is a k -dimensional T -orbit contained in $\bigcap_{i=1}^n \overline{D_i}$, the isotropy H in E is of the form

$$H = \{(t_1, \dots, t_{n-k}, 1, \dots, 1) \mid t_j \in \mathbb{C}^*\}.$$

Now the automorphism induced by the translation by

$$\lambda := (-1, \dots, -1) \in T$$

has no fixed points in $\Omega \cup \bigcup_{i=1}^n \bar{D}_i$ except p and is of order two. Thus it is a symmetry for $p \cdot \square$.

Remark:

If X is non-singular of dimension n , then exactly n T -stable divisors intersect in a fixed point of the T -action on X . In the singular case this number can be bigger.

So symmetries for the fixed points can be found very easily. On the other hand, if X is a compactification of an abelian group G , then the symmetries for points in $G \subset X$ are induced by the automorphism

$$g \mapsto g_0 g^{-1} \quad (g_0 \in G) \quad (\text{part I, Proposition 1.1}).$$

Therefore symmetries for the open orbit $\Omega = T$ of a complete normal torus embedding exist, if and only if the mapping

$$\psi : T \rightarrow T \quad \text{defined by} \quad t \mapsto t^{-1}$$

can be extended to an automorphism of X .

By Proposition 11 the mapping ψ is an automorphism of X , if and only if $\varphi = -\text{Id}$ is a fan automorphism, i.e.

$$\langle v_1, \dots, v_k \rangle \in \Sigma \Rightarrow \langle -v_1, \dots, -v_k \rangle \in \Sigma.$$

Such a fan is called centrally symmetric (cf. [V-K]).

So we have

Lemma 21:

Let X be a complete normal torus embedding with open orbit $\Omega = T$. Then there exist symmetries for $x \in \Omega$, if and only if the associated fan is centrally symmetric. $\square$

The only complete normal torus embedding in dimension one is $X = \mathbb{P}_1(\mathbb{C})$. Of course its fan is centrally symmetric (see Example above), and we know that X is a complex-symmetric torus embedding. But for $n > 1$, the variety $\mathbb{P}_n(\mathbb{C})$ is not a complex-symmetric torus embedding since the associated fan is not centrally symmetric. However, $\mathbb{P}_n(\mathbb{C})$ is complex-symmetric with respect to its group of automorphisms.

For a centrally symmetric fan we have $v \in \text{Sk}(\Sigma) \Rightarrow -v \in \text{Sk}(\Sigma)$.

Therefore $\{v \mid v \in \text{Sk}(\Sigma)\}$ is even. Thus

Corollary:

The number of T-stable divisors of a complex-symmetric torus embedding is even. $\square$

The following Lemma is important for inductive arguments.

Lemma 22:

Let X be a complete smooth complex-symmetric torus embedding and E a T-orbit in X .

Then $\bar{E}$ is again a complete smooth complex-symmetric torus embedding via the symmetries for the ambient space.

Proof:

By Propositions 13 and 18 the variety $\bar{E}$ is a complete smooth torus embedding.

So we only have to show that for $x \in \bar{E}$ the symmetry s_x stabilizes $\bar{E}$. But by faithful linearization the differential

$d_x s_x : T_x(X) \rightarrow T_x(X)$ is $-\text{Id}$, since s_x is of order two and has isolated fixed points. Thus $d_x s_x$ stabilizes any linear subspace of $T_x(X)$, e.g. $T_x(\bar{E})$, and therefore s_x stabilizes $\bar{E}$. $\square$

Remark:

The above argument makes use of the fact that a neighbourhood of the origin in $T_x(X)$ can be mapped biholomorphically onto a neighbourhood of x in X . This is false in the singular case since the dimension of the Zariski tangent space is bigger than the dimension of X .

Now let G be a subgroup of $\text{Aut}_0(X)$ that contains all the symmetries of X .

Lemma 23:

If X is smooth and complex-symmetric, then G acts transitively on the set of fixed points of T .

Proof:

We proceed by induction on $n := \dim_{\mathbb{C}} X$.

For $n = 1$ the variety X is $\mathbb{P}_1(\mathbb{C})$, and the Lemma is true (see the Example above).

Now assume that the Lemma is true for $1 \leq k < n$.

If $\Omega = T$ is the open orbit, $X \setminus \Omega$ is connected and consists of finitely many T -stable divisors $D_1, \dots, D_s$.

Let $p \in D_1$ be an arbitrary fixed point of T and $D_1, \dots, D_n$ all divisors with $D_i \ni p$.

Since D_1 is symmetric with respect to a subgroup of G (Lemma 22), for a fixed point $p_i \in D_i$ there exists $g_i \in G$ with $g_i(p) = p_i$ (induction).

Therefore each fixed point $q \in D_1 \cup \dots \cup D_n$ can be written as $g(p)$ ($g \in G$).

Now this process can be repeated with another divisor. Since the set of divisors is finite and $X \setminus \Omega$ is connected the lemma is proved by induction. $\square$

Remark:

It will be shown later that in general G does not act transitively on the set of divisors.

It is clear that each symmetry of a complex-symmetric torus embedding defines a fan isomorphism of order two, i.e. an isomorphism $\varphi : N \rightarrow N$ with $\varphi^2 = \text{Id}$ such that the scalar extension

$$\begin{aligned} \varphi : N_{\mathbb{R}} &\rightarrow N_{\mathbb{R}} \text{ satisfies} \\ \varphi(\sigma) &\in \Sigma \quad \forall \sigma \in \Sigma. \end{aligned}$$

We now establish a necessary and sufficient condition for a fan to be the fan of a complex-symmetric torus embedding.

Theorem 24:

Let Σ be a non-singular complete fan in $N_{\mathbb{R}}$. Then Σ is the fan of a complex-symmetric torus embedding X , if and only if Σ fulfills the following condition:

For each cone $\sigma \in \Sigma$ there exists a fan isomorphism φ_{σ} of order two with $\varphi_{\sigma}|_{(\sigma)_{\mathbb{R}}} = \text{Id}$ such that for the induced isomorphism

$$\tilde{\varphi}_{\sigma} : N_{\mathbb{R}}/(\sigma)_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/(\sigma)_{\mathbb{R}} \text{ we have } \tilde{\varphi}_{\sigma} = \text{Id}$$

($(\sigma)_{\mathbb{R}}$ denotes the smallest linear subspace of $N_{\mathbb{R}}$ that contains σ).

Remark:

A fan satisfying the above condition will be called a complex-symmetric fan.

Proof:

(1) First assume that X is complete, smooth and complex-symmetric.

Let E be a T -orbit, $x \in E$, s_x a symmetry for x and φ the induced fan isomorphism of order two.

Let $\sigma = \langle v_1, \dots, v_k \rangle$ be the (simplicial) cone corresponding to E . Of course s_x stabilizes $\bar{E}$, thus $\varphi(\sigma) = \sigma$ and $(\sigma)_{\mathbb{R}}$ is stabilized. Since E is the open orbit of the torus embedding $\bar{E}$, a symmetry for $x \in E$ induces $-\text{Id}$ on the corresponding fan. By Proposition 18 this fan is the projection of $\text{St}(\sigma)$ on $N_{\mathbb{R}}/(\sigma)_{\mathbb{R}}$, so

$\tilde{\varphi} = -\text{Id}$, where $\tilde{\varphi}$ is the mapping $N_{\mathbb{R}}/(\sigma)_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/(\sigma)_{\mathbb{R}}$

induced by φ .

Now the linear independent vectors $v_1, \dots, v_k$ defining σ correspond to T -stable divisors $D_1, \dots, D_k$ with

$$\bar{E} = \bigcap_{i=1}^k D_i$$

Since $d_x s_x = -\text{Id}$ (here the smoothness of X is essential) the differential stabilizes each subspace of $T_x(X)$.

Therefore $T_x(D_i)$ is stabilized and by faithful linearization the symmetry s_x stabilizes D_i , hence $\varphi(\langle v_i \rangle) = \langle v_i \rangle$. Since v_i can be chosen as a primitive element of N we have

$$\varphi(v_i) = v_i, \text{ thus}$$

$$\varphi|_{(\sigma)_{\mathbb{R}}} = \text{Id}.$$

So the condition is necessary.

(2) We begin by showing that for a one-dimensional cone $\sigma = \langle v \rangle$ the fan isomorphism φ_{σ} induces a symmetry f_{σ} for the one-codimensional orbit E belonging to σ .

Since translations by elements of T induce the identity on Σ we can compose f_{σ} with translations without changing φ_{σ} . By Proposition 18 we know that $\bar{E}$ itself is a torus embedding for $T' := T/v$.

Note that $v \in N$ also denotes a one-parameter subgroup; this group acts trivially on $\bar{E}$.

By faithful linearization we have for $x \in E$

$$T_x(X) = T_x(\bar{E}) \oplus W,$$

where W is a one-dimensional subspace invariant under the action of $v \in N$.

Now consider the fan isomorphism φ_σ . Since $\varphi_\sigma(v) = v$ it follows that φ_σ stabilizes E . Moreover, $\tilde{\varphi} = -\text{Id}$. Thus an automorphism of the form $t' \mapsto t'_0 \cdot t'^{-1}$ ($t'_0, t' \in T'$) is induced on $T' \cong E$. Therefore it is clear that the induced automorphism f_σ has a fixed point x which is isolated in E and that $d_x f_\sigma|_{T_X(\bar{E})} = -\text{Id}$.

Now choose a basis of $T_X(X)$ corresponding to the decomposition

$$T_X(X) = T_X(\bar{E}) \oplus W \text{ of } T_X(X)$$

We then have

$$d_x f_\sigma = \begin{pmatrix} \boxed{-\text{Id}} & \begin{matrix} a_1 \\ \vdots \\ a_{n-1} \end{matrix} \\ 0 \dots 0 & s \end{pmatrix} \quad (a_i \in \mathbb{C}, s \neq 0)$$

Now the isotropy v contains an element t such that

$$d_x t = \begin{pmatrix} \boxed{\text{Id}} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \dots 0 & -1/s \end{pmatrix} \quad (v \text{ acts trivially on } T_X(\bar{E})).$$

The automorphism

$$\tilde{f}_\sigma := t \circ f_\sigma \text{ also induces } \varphi_\sigma \text{ on } \Sigma.$$

Since $\varphi_\sigma^2 = \text{Id}$, the automorphism $\tilde{f}_\sigma^2$ is a translation which fixes x , i.e. $\tilde{f}_\sigma^2 \in v$. Thus

$$d_x \tilde{f}_\sigma^2 = \text{Id},$$

and an easy calculation shows that $a_1 = \dots = a_{n-1} = 0$ and

$$d_x \tilde{f}_\sigma = -\text{Id}.$$

Again by faithful linearization of v we have $\tilde{f}_\sigma^2 = \text{Id}$. Since x is isolated fixed point the automorphism $\tilde{f}_\sigma$ is a symmetry.

(3) Now we can prove the sufficiency of the condition in the Theorem by induction.

Assume that for each complex-symmetric fan Σ there exist symmetries for the T -orbits of codimension $< k$.

Let E be an arbitrary T -orbit of codimension k and $\sigma \in \Sigma$ the corresponding cone. The cone σ is generated by $v_1, \dots, v_k \in N$

and $\langle v_i \rangle \in \Sigma$ defines a T -stable divisor D_i , where T acts with ineffectivity $v_i \in N$.

Now it is easy to see that the fan Σ_σ corresponding to $\overline{\text{orb}}(\sigma)$ is again complex-symmetric, because the fan isomorphisms φ_τ for $\tau \in \text{St}(\sigma)$ induce fan isomorphisms on Σ_σ with the required properties.

We know that D_1 is again a smooth torus embedding and that $E \subset D_1$ is a T -orbit of codimension $k - 1$ in D_1 . Since $\varphi_\sigma(v_1) = v_1$, the isomorphism φ_σ induces a fan isomorphism for $\Sigma_{\langle v_1 \rangle}$.

With the same arguments as above we see that each automorphism $f_\sigma : X \rightarrow X$ induced by φ_σ has a fixed point x isolated in E . By induction we can assume that the induced automorphism

$$f_\sigma : D_1 \rightarrow D_1$$

is a symmetry for $x \in D_1$, i.e.

$$f_\sigma^2|_{D_1} = \text{Id} \quad \text{and} \quad d_x f_\sigma|_{T_x(D_1)} = -\text{Id}.$$

Now combining f_σ with an element of v_1 does not change $f_\sigma|_{D_1}$ nor φ_σ . With the same arguments as above we can choose $t \in v_1$ such that

$$\tilde{f}_\sigma = t \circ f_\sigma$$

is a symmetry for $x \in E \subset X$.

Thus the sufficiency is proved by induction. $\square$

Remark:

Let X be a singular (but normal) complete complex-symmetric torus embedding and $E = \text{orb}(\sigma)$ a T -orbit in X . If s_x is a symmetry for $x \in E$, then s_x stabilizes $\bar{E}$, and on E (which is isomorphic to the quotient Γ' of T by the ineffectivity of the T -action) an automorphism of the form

$$t'_0 \rightarrow t'_0 t'^{-1} \quad (t'_0, t' \in T')$$

As in the proof of Lemma 21 we see that s_x induces a fan isomorphism φ of order two such that

$$\varphi(\sigma) = \sigma \quad (\text{But perhaps } \varphi|_{(\sigma)_{\mathbb{R}}} \neq \text{Id}).$$

and for the induced isomorphism

$$\tilde{\varphi} : N_{\mathbb{R}}/(\sigma)_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/(\sigma)_{\mathbb{R}} \quad \text{we have} \quad \tilde{\varphi} = \text{Id}.$$

So in the singular case we obtain a necessary condition for a fan to be the fan of a complex-symmetric torus embedding.

We now consider some examples in dimension two.

First it is known (cf. [Oda], p. 52) that a complete non-singular torus embedding of dimension two which cannot be blown down equivariantly to a manifold is either

$\mathbb{P}_1 \times \mathbb{P}_1$, $\mathbb{P}_2$ or a Hirzebruch surface F_a ($a > 1$; Examples in 1.2).

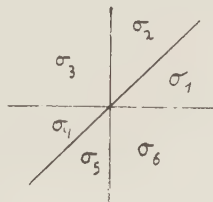
If we look at the corresponding fans we see that only $\mathbb{P}_1 \times \mathbb{P}_1$ has a centrally symmetric fan (cf. the example following Proposition 15) and that this fan satisfies the condition in Theorem 24. Therefore Theorem 24 and Lemma 21 imply

Corollary 25:

The only minimal complete complex-symmetric smooth torus embedding in dimension two is $\mathbb{P}_1 \times \mathbb{P}_1$. $\square$

We now give some non-minimal examples.

Consider the complete fan Σ with $\text{Sk}(\Sigma) = \{e_1, e_1+e_2, e_2, -e_1, -e_1-e_2, -e_2\}$. This defines a non-singular torus embedding. By Proposition 17 we see that X_Σ is obtained by blowing up $\mathbb{P}_1 \times \mathbb{P}_1$ in two suitable fixed points of T . The variety X_Σ is called a del-Pezzo surface.



The maximal-dimensional cones are denoted by σ_i ($i = 1, \dots, 6$).

Lemma 26:

The del-Pezzo surface is a complex-symmetric torus embedding.

Proof:

Since Σ is a centrally symmetric fan, by Theorem 24 we only have to prove that for each $v \in \text{Sk}(\Sigma)$ there exists a fan isomorphism φ_v such that

$$\varphi_v^2 = \text{Id}, \quad \varphi_v(v) = v \quad \text{and one eigenvalue of } \varphi_v \text{ is } -1.$$

For $v = e_1$, $v = -e_1$ we have the linear mapping

$$\varphi_v = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$$

which is of order two, fixes v and has an eigenvalue -1 . It is easy to see that φ_v induces a fan isomorphism (The skeleton of Σ is stabilized).

For $v = e_1 + e_2$, $v = -e_1 - e_2$ we can take the fan isomorphism

$$\varphi_v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{Z}),$$

and for $v = e_2$, $v = -e_2$ we can take

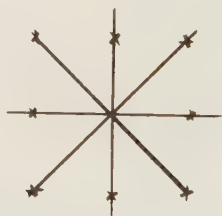
$$\varphi_v = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Thus Σ is complex-symmetric. $\square$

Remark:

It is easy to see that the group generated by φ_v ($v \in \text{Sk}(\Sigma)$) acts transitively on the set of one-dimensional cones. Thus the induced automorphisms act transitively on the set of divisors. Therefore all divisors have self-intersection number -1 .

If we blow up $\mathbb{P}_1 \times \mathbb{P}_1$ in the four fixed points of the standard T -action, the fan Σ corresponding to this variety X_Σ can be described as follows:



(the marked vectors are the elements of $\text{Sk}(\Sigma)$).

It is known that X_Σ is a complex-symmetric torus embedding ([Bo]). This can also be seen using Theorem 24:

$\varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ induces a symmetry for $\text{orb}(\langle e_1 \rangle)$ and $\text{orb}(\langle -e_1 \rangle)$.

$\varphi = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ induces a symmetry for $\text{orb}(\langle e_2 \rangle)$ and $\text{orb}(\langle -e_2 \rangle)$.

$\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ induces a symmetry for $\text{orb}(\langle e_1 + e_2 \rangle)$ and $\text{orb}(\langle -e_1 - e_2 \rangle)$.

$\varphi = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ induces a symmetry for $\text{orb}(\langle e_1 - e_2 \rangle)$ and $\text{orb}(\langle -e_1 + e_2 \rangle)$.

Remark:

In this example the group generated by the symmetries does not act transitively on the set of T-stable divisors but has two orbits:

$$A_1 := \{\overline{\text{orb}(\langle e_i \rangle)} \mid i = 1, 2\} \quad \text{and} \quad \{\overline{\text{orb}(\langle e_1 + e_2 \rangle)}\} =: A_2.$$

Indeed no automorphism of X_Σ can exchange elements of A_1 and A_2 since divisors in A_1 have self-intersection number -2 and divisors in A_2 have self-intersection number -1 .

This can be seen by applying the following general Lemma which is an immediate Corollary of Proposition 6.7 of [Oda].

Lemma 27:

Let X be a two-dimensional smooth complete torus embedding and Σ its corresponding fan.

If $n \in N$ defines a one-dimensional cone $\langle n \rangle$, then $\langle n \rangle$ is a face of two maximal-dimensional cones

$$\langle n_1, n \rangle \quad \text{and} \quad \langle n_2, n \rangle \quad (n_i \in N).$$

Moreover the number $\lambda \in \mathbb{Z}$ defined by

$$n_1 + n_2 = (-\lambda) \cdot n$$

is the self-intersection number of the divisor $\overline{\text{orb}(\langle n \rangle)}$. $\square$

This Lemma enables us to determine all complex-symmetric smooth torus embeddings in dimension two.

Indeed it will be shown that except for the three examples introduced above there is only one more example.

Now let Σ be a two-dimensional complex-symmetric fan which is not $\mathbb{P}^1 \times \mathbb{P}^1$. Since Σ is non-singular, vectors $n_1, n_2 \in N$ defining a two-dimensional cone in Σ can be chosen as a basis of N (Proposition 13).

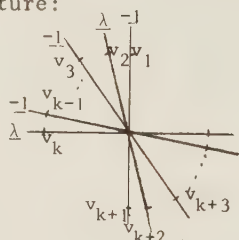
So we can assume that $\langle e_1, e_2 \rangle, \langle -e_1, -e_2 \rangle \in \Sigma$ (Lemma 21). Since the group generated by the symmetries acts transitively on the set of fixed points of T by Lemma 23, the corresponding fan isomorphisms act transitively on the set of two-dimensional cones in Σ .

Thus for $\langle v_1, v_2 \rangle \in \Sigma$ there exists a fan isomorphism φ such that

$$\varphi(\langle e_1, e_2 \rangle) = \langle v_1, v_2 \rangle \quad \text{and}$$

$$\begin{aligned} \text{either } \varphi(e_1) &= v_1 \quad \text{and} \quad \varphi(e_2) = v_2 \\ \text{or } \varphi(e_1) &= v_2 \quad \text{and} \quad \varphi(e_2) = v_1. \end{aligned}$$

Therefore the symmetries act on the set of T-stable divisors with at most two orbits. Consequently we have at most two different self-intersection numbers for the T-stable divisors in X_{Σ} . Since $X_{\Sigma} \cong \mathbb{P}_1 \times \mathbb{P}_1$, the variety X_{Σ} is not minimal (Corollary 25) and we have a divisor of self-intersection number -1 , say $\overline{\text{orb}(\langle e_2 \rangle)}$. If the self-intersection number of $\overline{\text{orb}(\langle e_1 \rangle)}$ is denoted by λ we have the following picture:



$\frac{i}{\lambda}$ denotes the self-intersection number of the divisor.

A direct application of Lemma 27 shows $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + v_{2k-1} = -\lambda \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since the first coordinate of v_{2k-1} is not negative we see $\lambda \leq 0$.

Lemma 28:

For the self-intersection number λ of $\overline{\text{orb}(\langle e_1 \rangle)}$ we have

$$-3 \leq \lambda \leq 0.$$

Proof:

The vector $v_{2\ell+1}$ ($0 \leq \ell \leq k-1$) defines a T-stable divisor with self-intersection number -1 , consequently by Lemma 27

$$v_{2\ell+1} = v_{2\ell} + v_{2\ell+2}.$$

By Proposition 17 this divisor can be blown down and the corresponding fan is obtained by replacing the cones

$$\langle v_{2\ell+1} \rangle, \langle v_{2\ell+1}, v_{2\ell} \rangle, \langle v_{2\ell+1}, v_{2\ell+2} \rangle \text{ by the cone } \langle v_{2\ell+2}, v_{2\ell} \rangle.$$

This process does not change the self-intersection numbers of $\overline{\text{orb}(\langle v_j \rangle)}$ ($j \neq 2\ell, 2\ell+1, 2\ell+2$).

A simple calculation shows that the self-intersection number of $\overline{\text{orb}(\langle v_{2\ell} \rangle)}$ and $\overline{\text{orb}(\langle v_{2\ell+2} \rangle)}$ is $\lambda + 1$ ($\neq -1$, if $\lambda \leq -3$), while the other self-intersection numbers do not change.

So we can blow down all divisors with self-intersection number -1 one after the other and we finally obtain a variety $X_{\Sigma'}$ such that

$$\text{Sk}(\Sigma') = \{v_{2i} \mid i = 1, \dots, k\}$$

and all self-intersection numbers are $\lambda + 2$.

Thus for $\lambda \leq -4$ they are all ≤ -2 , and $X_{\Sigma}^{\sim}$ cannot be blown down equivariantly to a manifold.

It follows that $X_{\Sigma}^{\sim}$ is $\mathbb{P}_1 \times \mathbb{P}_1$, $\mathbb{P}_2$ or a Hirzebruch-surface. But none of these varieties has only T-stable divisors with negative self-intersection number.

Therefore $\lambda \geq -3$. $\square$

The case $\lambda = 0$ can be excluded:

If $\lambda = 0$, then $v_{2k-1} = -e_2$ and $\langle e_1, -e_2 \rangle \in \Sigma$.

But in this case Σ cannot be centrally symmetric.

If $\lambda = -1$, we have $v_{2k-1} = -v_2$ by Lemma 27 and therefore $k = 3$.

The corresponding fan is isomorphic to the fan of the del-Pezzo surface (Apply the linear transformation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on Σ).

In the case $\lambda = -2$ we have $v_{2k-1} = \begin{pmatrix} -\lambda \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ by Lemma 27.

Since v_{2k-1} has self-intersection -1 we obtain (again by Lemma 27):

$$v_{2k-2} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -v_2.$$

Thus $k = 4$ and $Sk(\Sigma) = \{\pm e_1, \pm e_2, \pm(2e_1 - e_2), \pm(e_1 - e_2)\}$.

If we apply the linear transformation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on Σ , we obtain the fan corresponding to $\mathbb{P}_1 \times \mathbb{P}_1$ blown up in four points.

It remains to consider the case $\lambda = -3$.

The same calculations as above show

$$v_{2k-1} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad v_{2k-2} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad v_{2k-3} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \text{and} \quad v_{2k-4} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -v_2.$$

Thus $k = 6$ and

$$Sk(\Sigma) = \{\pm e_1, \pm e_2, \pm(3e_1 - e_2), \pm(2e_1 - e_2), \pm(3e_1 - 2e_2), \pm(e_1 - e_2)\}.$$

By a direct application of Theorem 24 we can see that Σ is a complex-symmetric fan. For convenience we list the fan isomorphisms:

$$\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \text{ for } \pm(3e_1 - e_2),$$

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \text{ for } \pm(2e_1 - e_2),$$

$$\begin{pmatrix} -1 & -3 \\ 0 & 1 \end{pmatrix} \text{ for } \pm(3e_1 - 2e_2),$$

$$\begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} \text{ for } \pm(e_1 - e_2),$$

$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \text{ for } e_1,$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \text{ for } e_2.$$

Thus X_Σ is a complex-symmetric variety.

We see that the divisors belonging to

$$L(3e_1 - e_2) \quad \text{and} \quad L(3e_1 - 2e_2)$$

can be blown down equivariantly.

After blowing down these four divisors we obtain the fan of $\mathbb{P}_1 \times \mathbb{P}_1$ blown up in four points.

Thus we have proved:

Theorem 29:

There are exactly four non-isomorphic complete complex-symmetric smooth torus embeddings of dimension two:

- 1) $Y_1 := \mathbb{P}_1 \times \mathbb{P}_1$,
- 2) $Y_2 :=$ the del Pezzo-surface, i.e. $\mathbb{P}_1 \times \mathbb{P}_1$ blown up in two suitable fixed points of the standard $(\mathbb{C}^*)^2$ -action,
- 3) $Y_3 := \mathbb{P}_1 \times \mathbb{P}_1$ blown up in all four fixed points of the standard $(\mathbb{C}^*)^2$ -action on $\mathbb{P}_1 \times \mathbb{P}_1$,
- 4) $Y_4 := Y_3$ blown up in four suitable fixed points of the $(\mathbb{C}^*)^2$ -action

1.4. Complex-symmetric Torus Embeddings and Coxeter Groups

In this paragraph we show that the skeleton of a complex-symmetric fan can be regarded as the root system of a Coxeter group.

A Coxeter group is a finite subgroup G of the orthogonal group $O(\mathbb{R}^n)$ which is generated by reflections $([G-B])$, i.e. orthogonal mappings of order two that stabilize a linear hyperplane pointwise and have an eigenvalue -1 (see [Bou] and [G-B], p. 125 for a different approach).

If we have a Coxeter group G then we can define the root system of G as follows (see [G-B] for details).

For each reflection S in G there are two eigenvectors of norm 1 for the (-1) -eigenvalue of S (instead of unit vectors sometimes vectors of arbitrary length are considered). These are the roots belonging to S .

The set of all roots is called the root system Δ of G . By construction G acts on Δ , but in general not transitively.

A base of a root system is a set of linear independent roots such that each element of Δ can be written as a linear combination of the base vectors with all coefficients either nonnegative (positive roots) or nonpositive (negative roots).

The reflections corresponding to a fixed base $r_1, \dots, r_n \in \Delta$ are called fundamental reflections and are denoted by S_{r_i} ($i = 1, \dots, n$). The group G is generated by the reflections S_{r_i} .

A Coxeter group is called irreducible, if its base cannot be written as union of two non-empty orthogonal subsets.

Equivalently, an irreducible Coxeter group cannot be written as a direct product of proper subgroups which are Coxeter groups.

If we have chosen a fixed base π of Δ , we can associate a Coxeter-graph (i.e. a marked graph with some additional properties) as follows:

To each $r_i \in \pi$ we associate a node i .

The nodes j and i are joined by a line and marked with the integer p_{ij} , if $S_{r_j} \circ S_{r_i}$ has order p_{ij} . If the order of $S_{r_j} \circ S_{r_i}$ is two, then the two nodes are not joined. In this way we obtain a positive definite Coxeter graph.

It can be proved:

- (1) Two Coxeter groups having the same Coxeter graph are conjugate to each other by an orthogonal transformation.
- (2) The Coxeter group is irreducible, if and only if its Coxeter graph is connected.

Positive definite Coxeter graphs are completely classified and it is shown that each positive definite Coxeter graph defines a Coxeter group ([G-B], Chapter 5.3).

Thus Coxeter groups are completely classified (at first in [Cox]).

The explicit knowledge of all Coxeter groups will be used in the sequel to classify all smooth symmetric torus embeddings. But before proving the classification we have to establish a link between complex-symmetric fans and Coxeter groups.

This will be done in a sequence of Lemmas.

Lemma 30:

Let Σ be a fan and G be the group of all fan isomorphisms. Then G is a finite group.

Proof:

If φ is a fan isomorphism, then φ permutes the elements of $Sk(\Sigma)$, because the vectors in $Sk(\Sigma)$ are primitive elements.

Of course a fan isomorphism which acts trivially on $Sk(\Sigma)$ is trivial. Therefore G is a subgroup of the permutation group of the elements of $Sk(\Sigma)$. Since Σ is complete, $Sk(\Sigma)$ is finite. Thus G is finite. $\square$

Remark:

If we have a finite subgroup G of $GL(n, \mathbb{R})$, we can always assume $G < O(\mathbb{R}^n)$ (orthogonal group) by choosing a suitable inner product. If $(.,.)$ denotes the canonical inner product, define

$$[x, y] := \sum_{g \in G} (g(x), g(y)) .$$

Then $[.,.]$ is an inner product and each $g \in G$ is orthogonal relative to $[.,.]$.

By choosing an orthonormal basis relative to $[.,.]$ we can assume that $[.,.]$ is the canonical inner product.

(Caveat: This change of coordinates perhaps destroys the property that the vectors $v \in Sk(\Sigma)$ are primitive elements of $N \approx \mathbb{Z}^n$).

From now on we assume that the group G of all fan isomorphisms is a subgroup of the orthogonal group $O(\mathbb{R}^n)$.

Let Σ be a complex-symmetric fan.

By definition for each $v \in \text{Sk}(\Sigma)$ there exists $\varphi_v \in G$ such that $\varphi_v(v) = v$ and $\tilde{\varphi}_v = -\text{Id}$ ($\tilde{\varphi}_v : \mathbb{R}^n/(v)_{\mathbb{R}} \rightarrow \mathbb{R}^n/(v)_{\mathbb{R}}$). Since Σ is centrally symmetric, we have $-v \in \text{Sk}(\Sigma)$ and $\varphi_{-v} = \varphi_v$.

Now the orthogonal transformation

$$\hat{\phi}_v := \varphi_v \circ (-\text{Id}) \in O(\mathbb{R}^n)$$

stabilizes a hyperplane pointwise and has one eigenvalue -1 .

Therefore $\hat{\phi}_v$ is a reflection and the roots belonging to $\hat{\phi}_v$ are $\pm v/\|v\|$.

So if we define G as the subgroup of G generated by $\hat{\phi}_v$ ($v \in \text{Sk}(\Sigma)$), the group G is a finite subgroup of $O(\mathbb{R}^n)$ generated by reflections.

Thus we have proved

Lemma 31:

The group G generated by $\hat{\phi}_v$ ($v \in \text{Sk}(\Sigma)$) is a Coxeter group with root system

$$\Delta = \left\{ \frac{v}{\|v\|} \mid v \in \text{Sk}(\Sigma) \right\}. \quad \square$$

Remarks:

- (1) It is not clear that all vectors of $\text{Sk}(\Sigma)$ have the same length, since a Coxeter group need not act transitively on its root system. But it is clear that if $\Delta^1 \subset \Delta$ is a G -orbit, all vectors $v \in \text{Sk}(\Sigma)$ with $\frac{v}{\|v\|} \in \Delta^1$ have the same length.
- (2) If Σ is a complete fan defining a singular complex-symmetric torus embedding, then for each $v \in \text{Sk}(\Sigma)$ there exists a fan isomorphism φ_v with the same properties as in the smooth case (see the Remark following Theorem 24). Therefore even in the singular case the mappings $\hat{\phi}_v$ generate a Coxeter group G with root system

$$\Delta = \left\{ \frac{v}{\|v\|} \mid v \in \text{Sk}(\Sigma) \right\}$$

(again we use that Σ is centrally symmetric; cf. Lemma 21).

The following Lemma shows that it is enough to consider irreducible Coxeter groups.

Lemma 32:

The Coxeter group G is irreducible if and only if Σ is irreducible (i.e. Σ is not the product $\Sigma' \times \Sigma''$ of lower-dimensional fans; cf. Proposition 15).

Proof:

Since the fan Σ is complete it is clear that G does not act trivially on any linear subspace of $V = N_{\mathbb{R}}$.

Now let the group G be reducible. By definition Δ is the union of (at least two) non-empty mutually orthogonal subspaces. Thus there exists a decomposition $V = V' \oplus V''$ (orthogonal) such that

$$\Delta = (\Delta \cap V') \cup (\Delta \cap V'').$$

Therefore

$$\text{Sk}(\Sigma) = (\text{Sk}(\Sigma) \cap V') \cup (\text{Sk}(\Sigma) \cap V'').$$

Since Σ is complete and non-singular it is reducible by Lemma 16. Now assume that Σ is reducible and non-singular. By Lemma 16 there exist non-trivial subspaces V', V'' such that

$$\text{Sk}(\Sigma) = (\text{Sk}(\Sigma) \cap V') \cup (\text{Sk}(\Sigma) \cap V'') \quad \text{and} \quad V = V' \oplus V''.$$

Therefore by Lemma 31

$$\Delta = (\Delta \cap V') \cup (\Delta \cap V'') =: \Delta' \cup \Delta''.$$

The Lemma is proved if Δ' and Δ'' are orthogonal to each other.

Let $\frac{v}{\|v\|} \in \Delta'$ and E_v the hyperplane which is pointwise stabilized by $\hat{\phi}_v$.

Then $\dim(V' \cap E_v) = \dim V' - 1$, since $v \in V'$.

Thus $V' = (V' \cap E_v) \oplus (v)_{\mathbb{R}}$ is stabilized by $\hat{\phi}_v$.

Since $\hat{\phi}_v$ stabilizes Δ we have

$$\hat{\phi}_v(\Delta') = \Delta' \quad \text{and} \quad \hat{\phi}_v(\Delta'') = \Delta''.$$

Moreover,

$$\hat{\phi}_v(V'') = V'' \quad \text{because} \quad \Delta'' \quad \text{contains a base of} \quad V''.$$

Now either $\dim(V'' \cap E_v) = \dim V''$ or $\dim(V'' \cap E_v) = \dim V'' - 1$, since E_v is a hyperplane. Assume that E_v does not contain V'' . Since $\hat{\phi}_v|_{V''}$ is orthogonal, the orthogonal complement of $E_v \cap V''$ in V'' which is a one-dimensional eigenspace for $\hat{\phi}_v$, is stabilized. Thus

This is impossible since $v \in V'$.

Thus $V'' \cap E_V = V''$, and V'' is orthogonal to v , since $\hat{\phi}_V$ is an orthogonal mapping.

Therefore V'' is orthogonal to each element of Δ' and thus the subspace V'' is orthogonal to V' . $\square$

§ 2. Classification of Complex-symmetric Torus Embeddings

2.1. Classification in Dimension Three

In Theorem 29 we showed that there exist only four non-isomorphic smooth complex-symmetric torus embeddings in dimension two.

We now will show that each three-dimensional smooth complex-symmetric torus embedding is the product of $\mathbb{P}_1$ and a two-dimensional complex-symmetric torus embedding.

Our main argument will be the description of three-dimensional non-singular fans by triangulations of S^2 ([Oda], pp. 61 - 66).

Let X_Σ be a three-dimensional complete non-singular torus embedding, Σ its corresponding fan and $S \subset \mathbb{N}_{\mathbb{R}}$ a sphere (centered at the origin).

If we intersect Σ with S , we obtain a triangulation of S with spherical triangles

$$S = \bigcup_{\sigma \in \Sigma} (\sigma \cap S) \quad (\text{here the completeness of } \Sigma \text{ is important}).$$

Each vertex of the triangulation is obtained as

$$\langle v \rangle \cap S, \text{ where } v \in \text{Sk}(\Sigma).$$

Similarly, each spherical edge of the triangulation corresponds to a two-dimensional cone and each spherical triangle to a three-dimensional cone.

Remark:

Up to here we only need that the fan Σ is simplicial. In the singular case the intersection of the maximal-dimensional cones with S is a spherical polygon.

Definition:

Consider a triangulation of S and let p be a vertex of the triangulation. Then the number of spherical edges of the triangulation which contain p is denoted by

$$\text{Val}(p).$$

If $\text{Val}(p) = n$ we call p an n -valent vertex.

Remarks:

- (1) Of course $\text{Val}(p) \geq 3$ for each vertex.
- (2) If we consider a triangulation induced by a non-singular complete fan, then there exists exactly one $v \in \text{Sk}(\Sigma)$ such that $p = \langle v \rangle \cap S$.
Then $\text{Val}(p)$ is the number of two-dimensional cones in Σ which contain $\langle v \rangle$.

Lemma 33:

Let the triangulation of S be induced by a complex-symmetric fan Σ .

- (1) Then $\text{Val}(p)$ is even for each vertex p .
- (2) There exist integers n_1, n_2, n_3 (depending on Σ) such that for each vertex p
$$\text{Val}(p) = n_i \quad (\text{for a suitable } i) .$$

Proof:

- (1) Let $p = \langle v \rangle \cap S$ ($v \in \text{Sk}(\Sigma)$).
Then $D := \overline{\text{orb}(\langle v \rangle)}$ is a divisor in X_Σ . By Lemma 22, D is again complex-symmetric and by Proposition 18 the two-dimensional cones containing $\langle v \rangle$ correspond to the T -stable divisors in D . But by the Corollary of Lemma 21 the number of divisors of a complex-symmetric torus embedding is even. Thus $\text{Val}(p)$ is even.

- (2) Let $\sigma := \langle v_1, v_2, v_3 \rangle$ be a maximal-dimensional cone. Define
 $p_i := \langle v_i \rangle \cap S$ and
 $n_i := \text{Val}(p_i)$.

Let $p := S \cap \langle v \rangle$ be a vertex of the triangulation. The cone $\langle v \rangle$ lies in a maximal-dimensional cone $\langle v, w_1, w_2 \rangle$. Since Σ is complex-symmetric, by Lemma 23 there exists a fan isomorphism φ with

$$\varphi(\sigma) = \langle v, w_1, w_2 \rangle .$$

It follows that

$$\text{either } \varphi(v_1) = v \text{ or } \varphi(v_2) = v \text{ or } \varphi(v_3) = v$$

and consequently

$$\text{Val}(p) = n_i \text{ for suitable } i \in \{1, 2, 3\} . \quad \square$$

As a special case of the Dehn-Sommerville-equations ([Grü], p. 146, p. 425) we obtain the following

Lemma 34:

Let d be the number of vertices of a triangulation of S^2 . Then the number of spherical edges is $3d - 6$ and the number of spherical triangles is $2d - 4$. $\square$

Consequently if $d = \# Sk(\Sigma)$, it follows that $3d - 6$ is the number of two-dimensional cones in Σ and $2d - 4$ is the number of three-dimensional cones in Σ .

The following formula is proved directly from the Dehn-Sommerville-equations in dimension three.

Lemma 35 ([Oda], p. 75):

Consider a triangulation of S^2 .

For $n \geq 3$ define b_n as the number of n -valent vertices of the triangulation. Then

$$(35) \quad \sum_{n \geq 3} b_n (6 - n) = 12.$$

Proof:

Since $Val(p) \geq 3$ for each vertex p , we have that $\sum_{n \geq 3} b_n$ is the number d of all vertices. Moreover, each spherical edge contains two vertices. Thus $\sum_{n \geq 3} b_n \cdot n$ is twice the number of spherical edges which equals $3d - 6$ by the Dehn-Sommerville-equations. Therefore

$$\sum_{n \geq 3} b_n (6 - n) = 6 \cdot \sum_{n \geq 3} b_n - \sum_{n \geq 3} n b_n = 6d - 2(3d - 6) = 12,$$

and Formula (35) is proved.

Corollary 36:

If Σ is a complex-symmetric fan then there exists p (defined by $S \cap \langle v \rangle$) such that $Val(p) = 4$.

Proof:

For a complex-symmetric fan the number $Val(p)$ is even (Lemma 33). Hence $Val(p) \geq 4$ for each vertex p in the triangulation. If

$Val(p) > 4$ for each vertex, then $Val(p) \geq 6$ and

$$\sum_{n \geq 3} b_n (6 - n) \leq 0. \text{ This contradicts Formula (35).}$$

Thus there exists p with $Val(p) = 4$. $\square$

Lemma 37:

Let Σ be a complex-symmetric fan, $d := \#Sk(\Sigma)$ and $\langle v_1, v_2, v_3 \rangle \in \Sigma$. Define $p_i := \langle v_i \rangle \cap S$.

If $Val(p_1) = 2N$ and $Val(p_i) \neq 2N$ ($i = 2, 3$), then the number of $(2N)$ -valent edges in the triangulation is $\frac{2d-4}{2N}$.

Proof:

By Lemma 33 and by our assumption each spherical triangle contains exactly one vertex p with $Val(p) = 2N$.

On the other hand, by definition for a vertex p with $Val(p) = 2N$ there exist $2N$ spherical edges in the triangulation containing p . Consequently, there exist $2N$ spherical triangles containing p . So the map that associates to each spherical triangle the unique edge p with $Val(p) = 2N$ is a surjective map onto the set of $(2N)$ -valent edges, and each preimage consists of $2N$ spherical triangles.

Since the number of triangles is $2d - 4$, the set of $(2N)$ -valent edges contains $\frac{2d-4}{2N}$ elements. $\square$

The central result of this paragraph is the following.

Proposition 38:

Let Σ be a complex-symmetric fan, $d := \#Sk(\Sigma)$, $\langle v_1, v_2, v_3 \rangle \in \Sigma$ and $n_i := Val(p_i)$. Without loss of generality

$$4 = n_1 \leq n_2 \leq n_3.$$

Then there are only the following possibilities:

- (1) $n_1 = n_2 = n_3 = 4$ and $d = 6$
- (2) $n_1 = n_2 = 4$, $n_3 = 2N$ and $d = 2N + 2$ ($N > 2$)
- (3) $n_1 = 4$, $n_2 = n_3 = 6$ and $d = 14$
- (4) $n_1 = 4$, $n_2 = 6$, $n_3 = 8$ and $d = 26$
- (5) $n_1 = 4$, $n_2 = 6$, $n_3 = 10$ and $d = 62$.

Proof:

We know that $n_1 = 4$ by Corollary 36.

- (1) If $n_1 = n_2 = n_3 = 4$, then $b_4 = d$ and $b_n = 0$ ($n \neq 4$).

Thus by Formula (35): $d = 6$.

- (2) If $n_1 = n_2 = 4$ and $n_3 = 2N$ (n_1 has to be even), then by Lemma 37 the number of $(2N)$ -valent vertices is $\frac{2d-4}{2N} = \frac{d-2}{N}$.

Therefore the number of 4-valent vertices is $d - \frac{d-2}{N}$.

Formula (35) gives

$$12 = \frac{d-2}{N} (6-2N) + 2(d - \frac{d-2}{N}) \quad \Leftrightarrow d = 2N + 2.$$

Therefore in this case the number of $(2N)$ -valent vertices is two.

(3) We claim that $n_2 = 6$, if $n_1 = 4$ and $n_2 > 4$:

Since $n_3 \geq n_2 > 4$, the number of 4-valent vertices is $\frac{d}{2} - 1$ by Lemma 37.

If $n_2 > 6$, then

$$\begin{aligned} & b_{n_1}(6-4) + b_{n_2}(6-n_2) + b_{n_3}(6-n_3) \\ & \leq (\frac{d}{2} - 1) \cdot 2 - 2b_{n_2} - 2b_{n_3} = (\frac{d}{2} - 1) \cdot 2 - 2(\frac{d}{2} + 1) = -4 < 12. \end{aligned}$$

This contradicts (35).

Therefore $n_1 = 4$, $n_2 = 6$.

Now assume $n_3 = 6$, then $b_4 = \frac{d}{2} - 1$, $b_6 = \frac{d}{2} + 1$ and again by (35) we have $d = 14$.

(4) Now let $n_1 = 4$, $n_2 = 6$ and $n_3 = 2N > 6$. By Lemma 37

$$b_{n_1} = \frac{d}{2} - 1, \quad b_{n_2} = \frac{2d-4}{6}, \quad b_{n_3} = d - (\frac{d}{2} - 1) - (\frac{2d-4}{6}).$$

Then Lemma 37 implies that

$$12 + 10N = (6-N) \cdot d.$$

Obviously the only solutions for which d is integral, positive and $2N > 6$ are

$N = 3$ and $d = 14$ (this is the case (3)),

$N = 4$ and $d = 26$ (this is the case (4)) and

$N = 5$ and $d = 62$ (this is the case (5)). $\square$

We now show that some of the cases mentioned in Proposition 38 do not define complex-symmetric fans.

Lemma 39:

There are no 3-dimensional complex-symmetric fans with

$$d = 26 \quad \text{or} \quad d = 62.$$

Proof:

By Lemma 31 we know that d is the number of vectors of a three-dimensional root system.

If the root system is irreducible, then there are only the following possibilities ([G-B], p. 76):

G	$ \Delta $
A_3	12
B_3	18
J_3	30

If the root system is reducible, it has to be the product of the one-dimensional and a two-dimensional root system. By Theorem 29 the only possibilities for a two-dimensional root system $\tilde{\Delta}$ to define a complex-symmetric variety are

$$|\tilde{\Delta}| = 4, 6, 8, 12.$$

Therefore $|\Delta| = 2 + |\tilde{\Delta}| = 6, 8, 10, 14$.

This proves the Lemma. $\square$

Lemma 40:

There is no complex-symmetric fan with $d = 14$ which defines a 6-valent vertex.

Proof:

According to the proof of Lemma 39 we have $d = |\Delta| = 14$, if and only if G is reducible.

If G is reducible, we have

$$G \cong A_1 \times G_2 \cong A_1 \times H_2^6 \quad ([G-B], p. 76).$$

Let $v \in \text{Sk}(\Sigma)$ define a root belonging to A_1 . Then the corresponding vertex of the triangulation has to be 12-valent.

But if we have a 6-valent vertex, by Proposition 38, 12-valent vertices cannot exist. $\square$

Remark:

It is also possible to exclude these cases by direct calculations. But the proofs here show how simple the arguments are, if one uses the classification of Coxeter groups.

For the remaining cases we need the following Lemma.

Lemma 41:

If G is an irreducible Coxeter group of dimension n acting on $\mathbb{R}^n$, then there exist no non-trivial G -invariant subspaces.

Proof:

Let V be an invariant subspace, $k := \dim_{\mathbb{R}} V$ ($0 < k < n$) and S_V a reflection which stabilizes the hyperplane E_V pointwise. If $V \cap E_V = V$, then $v \in V^\perp$, since S_V is orthogonal.

If $\dim(V \cap E_V) = k - 1$, then there exists a one-dimensional subspace of V which is stabilized by S_V and on which S_V does not act trivially.

It follows $V = (V \cap E_V) \oplus (v)_{\mathbb{R}}$.

Thus for each reflection S_V either $v \in V^\perp$ or $v \in V$.

Therefore the group G is reducible. $\square$

We are now in a position to prove

Theorem 42:

Let X_Σ be a three-dimensional complex-symmetric torus embedding which is smooth and complete.

Then X_Σ is the product of $\mathbb{P}_1$ and a smooth two-dimensional complex-symmetric torus embedding.

Proof:

We use the notation of Proposition 38. By Lemma 39 and Lemma 40 we only have to consider the cases

$$n_1 = n_2 = n_3 = 4 ; d = 6 \text{ and}$$

$$n_1 = n_2 = 4 , n_3 = 2N ; d = 2N + 2 \text{ (} N > 2 \text{)} .$$

In the first case the fan is isomorphic to that of $(\mathbb{P}_1)^3$. To see this note that, since the fan is non-singular, we can assume that $\langle e_1, e_2, e_3 \rangle$ is a cone in Σ . Since Σ is centrally symmetric, we have $\langle -e_1, -e_2, -e_3 \rangle \in \Sigma$.

Now $d = 6$ and therefore $Sk(\Sigma) = \{\pm e_1, \pm e_2, \pm e_3\}$.

The completeness of Σ yields that Σ is the fan corresponding to $(\mathbb{P}_1)^3$.

In the second case the number of $(2N)$ -valent vertices is exactly two (see Proposition 38).

If $v \in Sk(\Sigma)$ defines a $(2N)$ -valent vertex, then the vertex defined by $-v$ is also $(2N)$ -valent.

Therefore for each fan isomorphism φ it follows that

$$\varphi(v) = \pm v,$$

and $(v)_{\mathbb{R}}$ is an invariant subspace of the Coxeter group G . Thus G is reducible by Lemma 41. Consequently X_{Σ} is the product of $\mathbb{P}_1$ and a smooth two-dimensional complex-symmetric torus embedding (Lemma 32). $\square$

Corollary 43:

The only minimal complex-symmetric torus embedding X_{Σ} of dimension three is $(\mathbb{P}_1)^3$.

Proof:

By Theorem 42 we have

$$X_{\Sigma} \cong \mathbb{P}_1 \times X_{\tilde{\Sigma}}.$$

Since X_{Σ} is minimal, it follows that $X_{\tilde{\Sigma}}$ is also minimal. Therefore by Theorem 29

$$X_{\tilde{\Sigma}} \cong \mathbb{P}_1 \times \mathbb{P}_1. \quad \square$$

Remark:

In the three-dimensional classification we mainly used properties of triangulations of S^2 and not much about the theory of Coxeter groups. Indeed all arguments involving Coxeter groups could be replaced by simple calculations.

In higher dimensions this approach does not work:

For example in dimension four the Dehn-Sommerville-equations can be stated as follows:

Let f_i be the number of i -dimensional simplices.

Then $f_2 = 2f_1 - 2f_0$ and $f_3 = f_1 - f_0$ ([Grü], p. 425) and

$$f_1 \leq \frac{1}{2} (f_0 - 1) \cdot f_0 \quad ([Grü], p. 173).$$

Thus f_1 cannot be calculated from f_0 as in the three-dimensional case.

Moreover, an analogue of Formula (35) is obtained as follows:

Let b_n be the number of spherical edges containing exactly n 4-simplices, then $\sum_{n \geq 3} b_n = f_1$.

Each 4-simplex contains 6 spherical edges. Hence

$$\sum b_n \cdot n = 6 \cdot f_3 = 6(f_1 - f_0) \quad \text{and thus}$$

$$\sum_{n \geq 3} b_n (6 - n) = 6f_0.$$

Since the last equation depends on f_0 , the arguments are necessarily much more complicated. In higher dimensions we will therefore make extensive use of the explicit knowledge of the root system of an irreducible Coxeter group.

Since in dimension four there are some exceptions, this case has to be treated separately from the general case.

2.2. Classification in Dimension Four

If we describe the skeleton $\text{Sk}(\Sigma)$ of a non-singular complex-symmetric fan by the root system of a Coxeter group, some information is lost. For example it is not clear how the two-, three- and four-dimensional cones look like. In general we need to carry out somewhat difficult calculations.

This can be avoided if one uses inductive arguments.

Let $v \in \text{Sk}(\Sigma)$, G the group of fan isomorphisms and G the Coxeter group associated to Σ as in Lemma 31.

We can assume that each element of G is orthogonal (with respect to the canonical inner product of $\mathbb{R}^n$; see Remark after Lemma 30). By Lemma 22 we know that the divisor D defined by $\overline{\text{orb}(\langle v \rangle)}$ is also complex-symmetric and that the symmetries for D are induced by the symmetries of X_Σ . Here the smoothness of X_Σ is essential. Therefore the fan isomorphisms for $\Sigma_{\langle v \rangle}$, which define the symmetries, are induced by fan automorphisms of Σ stabilizing v . Now we can obtain the fan corresponding to D if we project all cones in $\text{St}(\langle v \rangle)$ orthogonally on $(v)_{\mathbb{R}}^\perp$. The induced symmetries are orthogonal with respect to the canonical inner product and $\text{Sk}(\Sigma_{\langle v \rangle})$ again defines a root system in dimension $n - 1$ for a Coxeter group of dimension $n - 1$.

Since each fan isomorphism which stabilizes v also stabilizes $\text{St}(\langle v \rangle)$, we have the following fact.

Lemma 44:

Let Σ be a non-singular complex-symmetric fan and G the Coxeter group belonging to Σ .

Then for $r \in \Delta$ the stabilizer subgroup $\text{Stab}_r(G) < G$ acts on the root system belonging to the complex-symmetric variety $\overline{\text{orb}(\langle r \rangle)}$ as a group of orthogonal transformations. $\square$

Since for $r \in \Delta$ the group $\text{Stab}_r(G)$ can be calculated and the possible root systems for $\overline{\text{orb}(\langle r \rangle)}$ are known by induction, Lemma 44 will enable us to obtain enough information about the complex-symmetric fan Σ .

We now want to describe the groups $\text{Stab}_r(G)$.

The proof of the following Proposition can be found in ([G-B], p. 77).

Proposition 45:

Let Δ be an irreducible root system, $\pi := \{r_1, \dots, r_n\}$ a base of Δ and $S_i := S_{r_i}$ the fundamental reflections with respect to π .

Let $\pi^* := \{s_1, \dots, s_n\}$ be the dual basis to π and

$$H_i := \text{Stab}_{S_i}(G).$$

Then $H_i = \langle S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n \rangle$, i.e. H_i is a Coxeter group in dimension $n - 1$. $\square$

If one looks at the list of irreducible Coxeter groups, there exist (among some exceptional classes) three infinite series of Coxeter groups.

They are often called classical groups (because they occur in the classification of simple Lie algebras as the root systems of the classical simple Lie algebras) and are denoted by A_n, B_n and D_n . For convenience we list their Coxeter graphs here:



Lemma 46:

Let G be an irreducible Coxeter group of dimension ≥ 4 with root system Δ , $G \not\cong A_n$.

Then there exists $r \in \Delta$ such that $\text{Stab}_r(G)$ is a Coxeter group. This Coxeter group is irreducible, if $G \not\cong D_n$.

If $G \cong D_4$, it is isomorphic to $A_1 \times A_1 \times A_1$.

If $G \cong D_n$ ($n > 4$), it is isomorphic to $A_1 \times D_{n-2}$.

Proof (following [G-B], pp. 79 - 80):

(1) For each Coxeter group there exists $r \in \Delta$ which is orthogonal to all but one root of a fixed base $r_1, \dots, r_n$.

Thus $r = \lambda \cdot s_i$ (for suitable $i \in \{1, \dots, n\}$ and $\lambda \in \mathbb{R}^*$) and

$$\text{Stab}_r(G) = \text{Stab}_{S_i}(G)$$

is a Coxeter group by Proposition 45.

This stabilizer can be calculated from its Coxeter graph.

(2) For the non-classical groups we find ([G-B], p. 80):

G	Index i	Cox. Graph of G	Cox. Graph of Stab	Stab $\approx$
I_3	2	$\begin{array}{c} 5 \\ 1-2-3 \end{array}$	$\begin{array}{c} 1-2 \end{array}$	$A_1 \times A_1$
I_4	4	$\begin{array}{c} 5 \\ 1-2-3-4 \end{array}$	$\begin{array}{c} 5 \\ 1-2-3 \end{array}$	I_3
F_4	4	$\begin{array}{c} 4 \\ 1-2-3-4 \end{array}$	$\begin{array}{c} 4 \\ 1-2-3 \end{array}$	B_3
E_6	1	$\begin{array}{c} 4 \\ 2-3-4-5-6 \\ \\ 1 \end{array}$	$\begin{array}{c} 2-3-4-5-6 \end{array}$	A_5
E_7	2	$\begin{array}{c} 3-4-5-6-7 \\ \\ 1 \end{array}$	$\begin{array}{c} 3-4-5-6-7 \\ \\ 1 \end{array}$	D_6
E_8	8	$\begin{array}{c} 3-4-5-6-7-8 \\ \\ 1 \end{array}$	$\begin{array}{c} 3-4-5-6-7 \\ \\ 1 \end{array}$	E_7

(3) Now consider $G \approx B_n$.

We can assume that

$$\Delta = \{\pm e_k, \pm e_i \pm e_j, 1 \leq j < i \leq n, 1 \leq k \leq n\}$$

is the root system and that

$$\pi = \{r_1, \dots, r_n\} = \{e_1, e_2 - e_1, \dots, e_n - e_{n-1}\}$$

is a base of the root system ([G-B], p. 71, p. 76).

Then $r := e_n$ is orthogonal to $r_1, \dots, r_{n-1}$. Therefore

$$\text{Stab}_r(G) = \text{Stab}_{S_n}(G)$$

and the Coxeter graph of the stabilizer is of the form

$$\begin{array}{c} 4 \\ 1-2-3 \dots n-1 \end{array}$$

Thus $\text{Stab}_{S_n}(G) \approx B_{n-1}$.

(4) The last case we have to consider is D_n .

We can assume that

$$\Delta = \{\pm e_i, \pm e_j, 1 \leq i < j \leq n\}$$

$$\pi = \{r_1, \dots, r_n\} = \{e_1 + e_2, e_2 - e_1, \dots, e_n - e_{n-1}\}$$

is a base for Δ ([G-B], p. 71, p. 76).

Then $r := e_{n-1} + e_n$ is orthogonal to all base vectors except r_{n-1} . Therefore $\text{Stab}_r(G)$ is a Coxeter group and the corresponding Coxeter graph has the form

$$\begin{array}{ccccccc} & & 0 & - & 0 & & \\ & \nearrow & 3 & 4 & \dots & n-2 & n \\ 1 & 0 & & & & & \end{array}$$

For $n = 4$ this becomes

$$\begin{array}{ccc} & 0 & \\ & \nearrow & \\ 1 & 0 & 4 \end{array}, \text{ thus } \text{Stab}_r(\mathcal{D}_4) \cong A_1 \times A_1 \times A_1,$$

for $n = 5$ we obtain

$$\begin{array}{ccc} & 0 & \\ & \nearrow & \\ 1 & 0 & 3 \end{array} \begin{array}{ccc} & 0 & \\ & \nearrow & \\ & 5 & \end{array}, \text{ thus } \text{Stab}_r(\mathcal{D}_5) \cong A_3 \times A_1 \cong \mathcal{D}_3 \times A_1$$

and for $n > 5$ it follows that $\text{Stab}_r(\mathcal{D}_n) \cong \mathcal{D}_{n-2} \times A_1$. $\square$

Lemma 47:

If $G \cong A_n$ ($n \geq 3$), then $\text{Stab}_r(G) \cong A_{n-2}$ for each $r \in \Delta$

Proof:

Since A_n acts transitively on its root system ([G-B], p. 78) all stabilizers are conjugate.

We can assume that

$$\pi := \{e_2 - e_1, e_3 - e_2, \dots, e_{n+1} - e_n\} = \{r_1, \dots, r_n\} \text{ ([G-B], p. 71)}.$$

The root system for A_n is considered in $\mathbb{R}^{n+1}$ and all vectors of Δ are orthogonal to $(1, \dots, 1) \in \mathbb{R}^{n+1}$.

Consider $r = e_2 - e_1$. Then the group H generated by $S_{r_3}, \dots, S_{r_n}$ is contained in $\text{Stab}_r(G)$. This is a Coxeter group isomorphic to A_{n-2} .

Now $|A_n| = (n+1)!$ ([G-B], p. 80) and $|\Delta| = n^2 + n$ ([G-B], p. 76). By the transitivity of the A_n -action on Δ

$$|\text{Stab}_r(A_n)| = \frac{(n+1)!}{n(n+1)} = (n-1)! = |A_{n-2}| = |H|.$$

Thus $\text{Stab}_r(A_n) \cong A_{n-2}$. $\square$

We now come to the main point of this chapter.

Theorem 48:

Let Σ be a complex-symmetric fan in dimension four. Then Σ is reducible.

Outline of the proof:

Consider the fan Σ and the corresponding root system Δ belonging to the irreducible Coxeter group G .

The group G is isomorphic to one of the groups A_4, B_4, D_4, I_4, F_4 .

By Lemma 46 we can choose $r \in \Delta$ such that

$\text{Stab}_r(G)$ is a Coxeter group (if $G \not\cong A_4$) or

$\text{Stab}_r(G)$ is isomorphic to A_2 (if $G \cong A_4$).

If we construct the fan $\Sigma_{\langle r \rangle}$ corresponding to $\overline{\text{orb}(\langle r \rangle)}$ by projecting the cones in $\text{St}(\langle r \rangle)$ orthogonally on $(r)_{\mathbb{R}}^{\perp}$, then $\text{Stab}_r(G)$ acts on $\Sigma_{\langle r \rangle}$ as a group of orthogonal transformations (Lemma 44).

By Theorem 42 the fan $\Sigma_{\langle r \rangle}$ is reducible, i.e. $\Sigma_{\langle r \rangle}$ is either the fan corresponding to $(\mathbb{P}_1)^3$ or product of a two-dimensional (irreducible) fan and the fan corresponding to $\mathbb{P}_1$.

Since the symmetries of $\overline{\text{orb}(\langle r \rangle)}$ are induced by symmetries of Σ which are orthogonal by Lemma 44, it follows that the irreducible factors of $\Sigma_{\langle r \rangle}$ are orthogonal to each other.

It is difficult to decide which vectors of Δ define cones in $\text{St}(\langle r \rangle)$. Thus we consider the projection of all vectors of Δ on $(r)_{\mathbb{R}}^{\perp}$.

We then determine which systems in this projection are stabilized by the group $\text{Stab}_r(G)$. In this way we are able to list all possibilities for $\text{St}(\langle r \rangle)$.

In each case we will then obtain a contradiction to the assumption that X_{Σ} is irreducible.

Since the proof of Theorem 48 is very technical, it is convenient to divide it into several steps (Lemma 49 - Lemma 66).

Lemma 49:

If $\text{Stab}_r(G)$ is an irreducible Coxeter group then $\Sigma_{\langle r \rangle}$ is the fan of $(\mathbb{P}_1)^3$.

Proof:

Assume that $\Sigma_{\langle r \rangle}$ is the product of a two-dimensional irreducible fan and the fan defining $\mathbb{P}_1$. Then the fan of $\mathbb{P}_1$ is stabilized by all fan isomorphisms (See the proof of Theorem 42). Therefore $\text{Stab}_r(G)$ stabilizes a one-dimensional subspace of $(r)_{\mathbb{R}}^{\perp}$. By Lemma 41 the group $\text{Stab}_r(G)$ is reducible. $\square$

Remark:

If $\Sigma_{\langle r \rangle}$ is not the fan of $(\mathbb{P}_1)^3$ and $\text{Stab}_r(G)$ is reducible, then this group stabilizes the $(\mathbb{P}_1)$ -factor of $\Sigma_{\langle r \rangle}$.

We now look explicitly at the different possibilities for G . Since some of our considerations can be used for the general classification we will consider the case of arbitrary n .

Consider $G \cong A_n$.

We may assume that

$\Delta := \{ \pm(e_i - e_j) \mid 1 \leq j < i \leq n+1 \}$ is the root system of G and that

$\pi := \{e_{i+1} - e_i \mid 1 \leq i \leq n\}$ is a base of Δ ([G-B], p. 71, p. 76).

(Note that this is a root system in $(1, \dots, 1)_{\mathbb{R}}^{\perp} \subset \mathbb{R}^{n+1}$).

Since A_n acts transitively on its root system we can choose the vector $r := e_2 - e_1$.

The proof of Lemma 47 shows that $\text{Stab}_r(G)$ is generated by the reflections

$$s_{e_4 - e_3}, \dots, s_{e_{n+1} - e_n}$$

The reflection $s_{e_i - e_j}$ exchanges the vectors e_i and e_j and stabilizes all vectors of the form e_k ($k \neq i, j$).

Lemma 50:

The orthogonal projection $\tilde{\Delta}$ of Δ on $(r)_{\mathbb{R}}^{\perp}$ consists of the vectors

$$\pm(e_i - e_j) \quad (3 \leq j < i \leq n+1)$$

and of the vectors

$$\pm \left[\frac{1}{2}(e_1 + e_2) - e_j \right] \quad (3 \leq j \leq n+1).$$

Proof:

The vectors of the form $\pm(e_i - e_j)$ are elements of Δ orthogonal to r .

Obviously the vectors of the form $\pm[\frac{1}{2}(e_1 + e_2) - e_j]$ are orthogonal to r .

We have

$$\frac{1}{2}(e_1 + e_2) - e_j + \frac{1}{2}(e_1 - e_2) = e_1 - e_j \in \Delta,$$

$$\frac{1}{2}(e_1 + e_2) - e_j - \frac{1}{2}(e_1 - e_2) = e_2 - e_j \in \Delta,$$

$$-\frac{1}{2}(e_1 + e_2) + e_j + \frac{1}{2}(e_1 - e_2) = -e_2 + e_j \in \Delta,$$

$$-\frac{1}{2}(e_1 + e_2) + e_j - \frac{1}{2}(e_1 - e_2) = -e_1 + e_j \in \Delta.$$

Therefore $\tilde{\Delta}$ is the orthogonal projection of all elements of Δ on $(r)_{\mathbb{R}}^{\perp}$. $\square$

We now show that the root system of A_4 cannot define a complex-symmetric fan. For this we prove two Lemmas which will also be useful in the general classification.

Lemma 51:

Assume $n \geq 4$ and $G \cong A_n$.

Then there exists no vector $v \in \tilde{\Delta}$ such that

$$g(v) = \pm v \quad \forall g \in \text{Stab}_r(G).$$

Proof:

The group $\text{Stab}_r(G)$ contains the reflection $S_{e_i - e_k}$ ($3 \leq i, k \leq n + 1$).

Consider a vector of the form $e_i - e_j \in \tilde{\Delta}$. Since $n \geq 4$ there exists k such that $3 \leq i, j, k \leq n + 1$ with e_i, e_j, e_k pairwise different.

Since

$$S_{e_i - e_k}(e_i - e_j) = e_k - e_j \text{ is linearly independent from } e_i - e_j \text{ and}$$

$$S_{e_i - e_j}(\frac{1}{2}(e_1 + e_2) - e_j) = \frac{1}{2}(e_1 + e_2) - e_i \quad (3 \leq i, j \leq n + 1; i \neq j)$$

is linearly independent from $\frac{1}{2}(e_1 + e_2) - e_j$, the Lemma is proved. $\square$

Remark:

The vector $e_1 + e_2$ is stabilized by every $g \in \text{Stab}_r(G)$, but it is not an element of $\tilde{\Delta}$.

Lemma 52:

Assume $n \geq 4$, $G = A_n$ and r as above. Then for each $v \in \tilde{\Delta}$ there exists $g \in \text{Stab}_r(G)$ such that $g(v)$ and v are neither linearly dependent nor orthogonal to each other.

Proof:

One can use the same symmetries as in the foregoing Lemma:

$$s_{e_i - e_k}(e_i - e_j) = e_k - e_j \text{ is not orthogonal to } e_i - e_j ;$$

$$s_{e_i - e_j}(\frac{1}{2}(e_1 + e_2) - e_j) = \frac{1}{2}(e_1 + e_2) - e_i \text{ is not orthogonal to } \frac{1}{2}(e_1 + e_2) - e_j. \square$$

Corollary 53:

The root system of A_4 cannot define a complex-symmetric fan.

Proof:

If $\Sigma_{\langle r \rangle}$ is the fan of $(\mathbb{P}_1)^3$, then $\Sigma_{\langle r \rangle}$ contains vectors v_1, v_2, v_3 which are mutually orthogonal such that

$$g(v_i) = \pm v_j \quad (1 \leq i, j \leq 3) \quad \forall g \in \text{Stab}_r(G) .$$

But by Lemma 52 the set $\tilde{\Delta}$ does not contain vectors with these properties.

If $\Sigma_{\langle r \rangle}$ is the product of $\mathbb{P}_1$ and an irreducible two-dimensional fan, then by the Remark following Lemma 49 there is a vector $v \in \tilde{\Delta}$ such that

$$g(v) = \pm v \quad \forall g \in \text{Stab}_r(G) .$$

This contradicts Lemma 51. $\square$

Now consider the case $G \cong B_n$.

In this case we may assume that

$$\Delta := \{\pm e_i \mid 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \mid 1 \leq j < i \leq n\}$$

is the root system of G and that

$$\pi := \{e_1, e_2 - e_1, \dots, e_n - e_{n-1}\} \text{ is a base of } \Delta \quad ([G-B], \text{ p. 71, p. 76}).$$

If we choose $r = e_n$, then $\text{Stab}_r(G) \cong B_{n-1}$ (by Proposition 45).

We consider the orthogonal projection $\tilde{\Delta}$ of Δ on $(r)_{\mathbb{R}}^\perp$.

Since vectors of the form $\pm e_j \pm e_n$ are equivalent to the vectors $\pm e_j$, we have

$$\tilde{\Delta} = \{\pm e_i \mid 1 \leq i \leq n-1\} \cup \{\pm e_i \pm e_j \mid 1 \leq j < i \leq n-1\} .$$

This is the root system of B_{n-1} in $\mathbb{R}^{n-1}$.

For the Lemmas 54 - 57 we will assume $G \cong B_n$ and r as above.

Lemma 54:

Assume $n \geq 4$.

Then for each vector of the form $v := \pm e_i \pm e_j \in \tilde{\Delta}$ there exists $g \in \text{Stab}_r(G)$ such that $g(v)$ is not orthogonal to v and $g(v)$ and v are linearly independent.

Proof:

The group $\text{Stab}_r(G)$ is generated by the reflections

$$S_{e_1}, S_{e_2 - e_1}, \dots, S_{e_{n-1} - e_{n-2}} \quad (\text{Proposition 45}).$$

We have

$$S_{e_i - e_k}(\pm e_i \pm e_j) = \pm e_k \pm e_j \quad (k \neq i, j; 1 \leq i, j, k \leq n-1),$$

and the vectors $\pm e_i \pm e_j$ and $\pm e_k \pm e_j$ are neither linearly dependent nor orthogonal. $\square$

Lemma 55:

If $G \cong B_4$, then $\{\pm e_1, \pm e_2, \pm e_3\} \subset \tilde{\Delta}$ is the only system that can define the skeleton of $\Sigma_{\langle r \rangle}$.

Proof:

By Lemma 49 the fan $\Sigma_{\langle r \rangle}$ must be the fan of $(\mathbb{P}_1)^3$.

Thus there exist mutually orthogonal vectors $v_1, v_2, v_3 \in \tilde{\Delta}$ such that

$$\{\pm v_1, \pm v_2, \pm v_3\} \text{ defines } \Sigma_{\langle r \rangle}.$$

Since this system has to be stabilized by $\text{Stab}_r(G)$ this is only possible for $\pm e_1, \pm e_2, \pm e_3$ (by Lemma 54).

This system is stabilized by the generating reflections of $\text{Stab}_r(G)$. $\square$

Lemma 56:

If $\langle e_1, \dots, e_{n-1} \rangle$ is a cone in $\Sigma_{\langle r \rangle}$, then

$$\langle e_1 + e_n, \dots, e_{n-1} + e_n, e_n \rangle \text{ is a cone in } \Sigma.$$

Proof:

The cone $\langle e_1, \dots, e_{n-1} \rangle \in \Sigma_{\langle r \rangle}$ is obtained as the orthogonal projection of a maximal dimensional cone $\sigma \in \text{St}(\langle e_n \rangle)$ on $(e_n)_{\mathbb{R}}^\perp$.

Of course $\langle e_n \rangle$ is a face of σ .

The vector e_i is obtained as an orthogonal projection of

$$e_i, e_i + e_n \text{ or } e_i - e_n \in \Delta.$$

Therefore either $\langle e_i \rangle$ or $\langle e_i - e_n \rangle$ or $\langle e_i + e_n \rangle$ is a face of σ .

If $\langle e_i - e_n \rangle$ is a face of σ , then $\langle e_i - e_n, e_n \rangle \in \Sigma$. But $\langle e_i \rangle \subset \langle e_i - e_n, e_n \rangle$ and $\langle e_i \rangle \in \Sigma$ is not a face of $\langle e_i - e_n, e_n \rangle$. This contradicts the properties of a fan.

If $\langle e_i \rangle$ is a face of σ , then $\langle e_i, e_n \rangle \in \Sigma$, but $\langle e_i + e_n \rangle \subset \langle e_i, e_n \rangle$, and we again obtain a contradiction.

Thus $\langle e_i + e_n \rangle$ must be a face of σ .

Consequently

$$\sigma = \langle e_1 + e_n, \dots, e_{n-1} + e_n, e_n \rangle. \quad \square$$

Lemma 57:

Assume $n \geq 4$.

If the root system Δ of B_n defines a fan Σ and if

$$\langle e_1 + e_n, \dots, e_{n-1} + e_n, e_n \rangle \in \Sigma,$$

then Σ is not complex-symmetric.

Proof:

Since $\langle e_1 + e_n, \dots, e_{n-1} + e_n, e_n \rangle \in \Sigma$, we have

$$\langle e_1 + e_n, \dots, e_{n-1} + e_n \rangle \in \Sigma.$$

Now Σ is complex-symmetric. Thus there exists a fan automorphism φ (which is orthogonal by Lemma 30 and by assumption) of order two such that

$$\varphi(e_i + e_n) = e_i + e_n \quad (1 \leq i \leq n-1) \quad \text{and}$$

$$\varphi(e_1 + \dots + e_{n-1} - e_n) = -e_1 - \dots - e_{n-1} + e_n$$

(this vector is orthogonal to $e_1 + e_n, \dots, e_{n-1} + e_n$).

We obtain

$$\varphi(e_1 + \dots + e_{n-1} + (n-1)e_n) = \varphi((e_1 + e_n) + \dots + (e_{n-1} + e_n)) = e_1 + \dots + e_{n-1} + (n-1)e_n$$

and

$$\varphi(e_1 + \dots + e_{n-1} + (n-1)e_n) = \varphi(e_1 + \dots + e_{n-1} - e_n + ne_n) = -e_1 - \dots - e_{n-1} + e_n + n\varphi(e_n)$$

Thus

$$2(e_1 + \dots + e_{n-1}) + (n-2)e_n = n\varphi(e_n)$$

and

$$\varphi(e_n) = \frac{2}{n}(e_1 + \dots + e_{n-1}) + \frac{n-2}{n}e_n.$$

For $n > 2$ no scalar multiple of this vector is in Δ .

Thus by Theorem 24, Σ cannot be complex-symmetric. $\square$

As an immediate consequence of the foregoing Lemmas we eliminate the case B_4 .

Corollary 58:

The root system of B_4 cannot define a complex-symmetric fan. $\square$

Now consider the case $G \cong \mathcal{D}_n$.

We may assume that

$\Delta := \{\pm e_i \pm e_j \mid 1 \leq j < i \leq n\}$ is the root system of G and that

$\pi := \{e_1 + e_2, e_2 - e_1, \dots, e_n - e_{n-1}\}$ is a base for Δ ([G-B], p. 71, p. 76).

If we define $r := e_{n-1} + e_n$, then $\text{Stab}_r(G) \cong \mathcal{D}_{n-2} \times A_1$ (resp. $A_1 \times A_1 \times A_1$, if $n = 4$) by Lemma 46, and this group is generated by the reflections

$s_{e_1 + e_2}, s_{e_2 - e_1}, \dots, s_{e_{n-2} - e_{n-3}}, s_{e_n - e_{n-1}}$

(the last reflection generates the A_1 -part of $\text{Stab}_r(G)$).

For the next two Lemmas we assume $G = \mathcal{D}_n$ and r as above.

Lemma 59:

Let $\tilde{\Delta}$ be the orthogonal projection of Δ on $(e_{n-1} + e_n)^\perp_{\mathbb{R}}$. Then

$$\begin{aligned} \tilde{\Delta} &= S_1 \cup S_2 \cup S_3 \\ &= \{\pm \frac{1}{2}(e_n - e_{n-1}) \pm e_i \mid 1 \leq i \leq n-2\} \cup \{\pm(e_{n-1} - e_n)\} \\ &\quad \cup \{\pm(e_i \pm e_j) \mid 1 \leq j < i \leq n-2\}. \end{aligned}$$

The set S_3 (resp. S_2) is the root system for $\mathcal{D}_{n-2}$ (resp. A_1).

Proof:

Of course all vectors of $\tilde{\Delta}$ are orthogonal to $e_{n-1} + e_n$.

The elements of S_2 and S_3 are vectors in Δ which are orthogonal to $e_{n-1} + e_n$.

For vectors in S_1 we have

$$\begin{aligned} \frac{1}{2}(e_n - e_{n-1}) \pm e_i + \frac{1}{2}(e_n + e_{n-1}) &= e_n \pm e_i \\ \frac{1}{2}(e_n - e_{n-1}) \pm e_i - \frac{1}{2}(e_n + e_{n-1}) &= -e_{n-1} \pm e_i \\ -\frac{1}{2}(e_n - e_{n-1}) \pm e_i + \frac{1}{2}(e_n + e_{n-1}) &= e_{n-1} \pm e_i \\ -\frac{1}{2}(e_n - e_{n-1}) \pm e_i - \frac{1}{2}(e_n + e_{n-1}) &= -e_n \pm e_i \end{aligned}$$

So $\tilde{\Delta}$ is the set of all projections of vectors in Δ . $\square$

For the general classification we need the following

Lemma 60:

If $n \geq 5$, then for each vector $v \in \tilde{\Delta} \setminus S_2$ there exists $g \in \text{Stab}_r(G)$ such that $g(v)$ is neither orthogonal to v nor linearly dependent of v .

Proof:

For a vector of the form $\pm e_i \pm e_j$ ($1 \leq j < i \leq n-2$) there exists $1 \leq k \leq n-2$, $k \neq i, j$ (Here $n \geq 5$ is important).

Thus $S_{e_i - e_k}(\pm e_i \pm e_j) = \pm e_k \pm e_j$ has the required properties.

For a vector of the form $\pm \frac{1}{2}(e_n - e_{n-1}) \pm e_i$ we can take $S_{e_i - e_k}$ ($1 \leq k \leq n-2$; $k \neq i$). $\square$

Remark:

Of course S_2 is stabilized by $\text{Stab}_r(G)$, because this is the root system belonging to A_1 .

Now we come back to the case $n = 4$.

For $n = 4$ the group $\text{Stab}_r(G)$ is generated by the reflections

$$S_{e_1 + e_2}, S_{e_1 - e_2}, S_{e_4 - e_3}$$

and we have

$$\tilde{\Delta} = \{\pm e_1 \pm e_2\} \cup \{\pm(e_3 - e_4)\} \cup \{\pm \frac{1}{2}(e_4 - e_3) \pm e_1\} \cup \{\pm \frac{1}{2}(e_4 - e_3) \pm e_2\}.$$

The group $\text{Stab}_r(G)$ acts with four orbits on $\tilde{\Delta}$:

$$\text{Stab}_r(G)(e_3 - e_4) = \{\pm(e_3 - e_4)\};$$

$$\text{Stab}_r(G)(e_1 + e_2) = \{\pm(e_1 + e_2)\};$$

$$\text{Stab}_r(G)(e_1 - e_2) = \{\pm(e_1 - e_2)\};$$

$$(\text{Hence it follows that } \text{Stab}_r(G) \cong A_1 \times A_1 \times A_1);$$

$$\text{Stab}_r(G)(e_1 \pm \frac{1}{2}(e_4 - e_3)) = \{e_i \pm \frac{1}{2}(e_4 - e_3) \mid i = 1, 2\}.$$

In order to determine all possibilities for $\Sigma_{<r>}$ we have to look at complete systems in $\tilde{\Delta}$ which are invariant under $\text{Stab}_r(G)$.

Since $\Sigma_{<r>}$ is a reducible fan it is easy to see that

$$\{+(e_1 + e_2), +(e_1 - e_2), +(e_3 - e_4)\}$$

is the only system in $\tilde{\Delta}$ which defines a three-dimensional complex-symmetric fan. In this case $\Sigma_{<r>}$ is the fan corresponding to $(\mathbb{W}_1)^3$.

Therefore $\langle c_1+c_2, c_1-c_2, c_3-c_4 \rangle$ is a cone in $\Sigma_{\langle r \rangle}$ and $\sigma := \langle c_1+c_2, c_1-c_2, c_3-c_4, c_3+c_4 \rangle$ is a cone in Σ .
Now σ contains the cone $\langle c_1+c_3 \rangle$, but this cone is not a face of σ . This contradicts the definition of a fan.
Thus we have proved

Lemma 61:

The root system of D_4 does not arise from a complex-symmetric fan. $\square$

For the case $G \cong I_4$ we need the following fact about the fan automorphisms of fans defining $(\mathbb{P}_1)^k$. The proof is immediate.

Lemma 62:

Let Σ be a fan defining $(\mathbb{P}_1)^k$. Then the group G of fan isomorphisms has order $2^k \cdot (k!)$. $\square$

Lemma 63:

The root system of $G \cong I_4$ cannot define a complex-symmetric fan.

Proof:

Let Δ be the root system of I_4 . By Lemma 46 we know that there exists $r \in \Delta$ such that

$$\text{Stab}_r(G) \cong I_3.$$

Since this group is irreducible the fan $\Sigma_{\langle r \rangle}$ is the fan of $(\mathbb{P}_1)^3$ (Lemma 49). Now, since $\Sigma_{\langle r \rangle}$ is a complete fan and thus its skeleton contains a basis of $(r)_{\mathbb{R}}^\perp$, an element of I_3 , which acts trivially on $\Sigma_{\langle r \rangle}$ is the identity.

So if I_3 stabilizes $\Sigma_{\langle r \rangle}$ its order is at most

$$2^3 \cdot 3! = 48 \text{ by Lemma 62.}$$

But the order of I_3 is 120 ([G-B], p. 80). $\square$

The remaining case in the classification is the case $G \cong F_4$. It turns out that this is the most difficult example. This is due to the fact that the properties for a complex-symmetric fan are "almost-fulfilled".

We may assume that

$$\begin{aligned} \Delta := S_1 \cup S_2 \cup S_3 = & \{ \pm e_i \mid 1 \leq i \leq 4 \} \cup \{ \pm e_i \pm e_j \mid 1 \leq j < i \leq 4 \} \\ & \cup \left\{ \frac{1}{2} \sum_{i=1}^4 \epsilon_i e_i \mid \epsilon_i = \pm 1 \right\} \end{aligned}$$

is the root system of F_4 and that

$$\pi := \{r_1, \dots, r_4\} = \{-\frac{1}{2}(e_1+e_2+e_3+e_4), e_1, e_2-e_1, e_3-e_2\}$$

is a base of Δ ([G-B], p. 71, p. 76).

In this case the matrices for the fundamental reflections are

$$S_{r_1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad S_{r_2} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S_{r_3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_{r_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

An easy calculation shows that F_4 acts with two orbits on its root system. We have

$$G(e_4) = S_1 \cup S_3 \quad \text{and}$$

$$G(e_3-e_4) = S_2,$$

and both orbits consist of 24 vectors ([G-B], p. 78).

The vector $r = e_4$ is orthogonal to r_2, r_3, r_4 but not orthogonal to r_1 . Proposition 45 shows that $\text{Stab}_r(G) \cong B_3$ and that this group is generated by $S_{r_2}, S_{r_3}, S_{r_4}$.

In this case the projection $\tilde{\Delta}$ of Δ on $(e_4)_{\mathbb{R}}^\perp$ consists of the vectors

$$\pm e_1, \pm e_2, \pm e_3, \pm e_1 \pm e_2, \pm e_1 \pm e_3, \pm e_2 \pm e_3$$

and

$$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3).$$

Note that e_i is obtained as the projection of the vectors

$$e_i, e_i - e_4, e_i + e_4 \quad (i = 1, 2, 3).$$

Since $\text{Stab}_r(G)$ is irreducible, the fan $\Sigma_{\langle r \rangle}$ is the fan of $(\mathbb{P}_1)^3$ by Lemma 49.

Lemma 64:

The only possibility for the skeleton of $\Sigma_{\langle r \rangle}$ is

$$\text{Sk}(\Sigma_{\langle r \rangle}) = \{\pm e_1, \pm e_2, \pm e_3\}.$$

Proof:

We must show that for each $r \in \tilde{\Delta} \setminus \{\pm e_1, \pm e_2, \pm e_3\}$ there exists $g \in \text{Stab}_r(G)$ such that $g(v)$ is not orthogonal and not linear dependent on v .

For $v = \frac{1}{2}(e_1 \pm e_2 \pm e_3)$ we have

$$S_{r_2}(v) = \frac{1}{2}(-e_1 \pm e_2 \pm e_3),$$

and these two vectors are neither orthogonal nor linearly dependent.

For $v = e_i \pm e_j$ ($i = 2, 3$) we can take S_{r_4} .

This mapping exchanges e_2 and e_3 .

For $v = e_2 \pm e_3$ we can take S_{r_3} which exchanges e_1 and e_2 .

Since S_{r_2}, S_{r_3} and S_{r_4} permute the vectors $\pm e_1, \pm e_2, \pm e_3$, this complete system is stabilized by $\text{Stab}_r(G)$. $\square$

Lemma 65:

The fan Σ contains a cone of the form

$$\langle e_4, e_1 + e_4, e_2 + e_4, e_3 + e_4 \rangle.$$

Proof:

Since $\langle e_1, e_2, e_3 \rangle$ is a cone in $\Sigma_{\langle r \rangle}$, $\langle e_1 \rangle$ is a cone in $\Sigma_{\langle r \rangle}$.

Thus $\text{St}(\langle e_4 \rangle)$ contains either a cone of the form $\langle e_1 - e_4, e_4 \rangle$

or of the form $\langle e_1, e_4 \rangle$ or of the form $\langle e_1 + e_4, e_4 \rangle$.

The first cone contains $\langle e_1 \rangle$, but not as a face.

The second cone contains $\langle e_1 + e_4 \rangle$, but not as a face.

So in these cases we obtain a contradiction to the definition of a fan. Thus the only possibility is $\langle e_1 + e_4, e_4 \rangle$.

Using the same arguments for e_2 and e_3 , it follows that

$$\langle e_1, e_2, e_3 \rangle \in \Sigma_{\langle r \rangle} \text{ is the projection of}$$

$$\langle e_1 + e_4, e_2 + e_4, e_3 + e_4, e_4 \rangle. \quad \square$$

The following Lemma shows that F_4 cannot define a complex-symmetric fan. However it turns out that there are very many symmetries for

$\langle e_1 + e_4, e_2 + e_4, e_3 + e_4, e_4 \rangle$ stabilizing the root system Δ .

Lemma 66:

If $\langle e_4, e_1 + e_4, e_2 + e_4, e_3 + e_4 \rangle \in \Sigma$, then Σ cannot be complex-symmetric.

Proof:

By definition of a complex-symmetric fan there exists a fan automorphism φ of order two such that

$$\varphi|_{(e_1+e_4, e_2+e_4)\mathbb{R}} = \text{Id}$$

and that

$$\varphi|_{(e_3, e_1+e_2-e_4)\mathbb{R}} = -\text{Id}.$$

The last condition follows from the fact that φ is orthogonal.

It follows that φ corresponds to the matrix

$$\varphi = \begin{pmatrix} 1/3 & -2/3 & 0 & 2/3 \\ -2/3 & 1/3 & 0 & 2/3 \\ 0 & 0 & -1 & 0 \\ 2/3 & 2/3 & 0 & 1/3 \end{pmatrix}.$$

Obviously this mapping does not stabilize the root system Δ of F_4 . $\square$

The above Lemma completes the proof of Theorem 48.

2.5. General classification of Complex-symmetric Torus Embeddings

The goal of this section is to prove

Theorem 67:

Let X be a smooth complete complex-symmetric torus embedding of dimension n . Then

$$X \cong X_{i_1} \times \dots \times X_{i_k} \times (\mathbb{P}^1)^{\ell} \quad (2k + \ell = n),$$

where X_{i_j} is a two-dimensional irreducible complex-symmetric torus embedding (cf. Theorem 29).

The proof will be divided in a sequence of Lemmas.

We proceed by induction on $n := \dim_{\mathbb{C}} X$.

By Theorems 29, 42 and 48 we may assume that the Theorem is proved for $\dim_{\mathbb{C}} X \leq 4$.

So let Σ be an irreducible complex-symmetric fan of dimension $n > 4$ (always assumed to be non-singular).

Let G be the irreducible Coxeter group associated to $\text{Sk}(\Sigma)$. It follows that G is one of the following groups:

$$A_n, B_n, D_n, E_6, E_7, E_8.$$

The root system of G is denoted by Δ .

By induction we may assume that each lower-dimensional complex-symmetric fan is the product of irreducible two- and one-dimensional fans. The irreducible factors are orthogonal to each other, if we choose a suitable inner product.

Now we know that for the irreducible root system Δ there exists $r \in \Delta$ such that either $\text{Stab}_r(G)$ is an irreducible Coxeter group in dimension $n - 1$ (if $G \not\cong A_n, D_n$) or $\text{Stab}_r(G)$ contains an irreducible Coxeter group in dimension $n - 2$ (if $G \cong A_n, D_n$; Lemmas 46 and 47).

By Lemma 44 the group $\text{Stab}_r(G)$ acts on the fan $\Sigma_{\langle r \rangle}$ corresponding to the complex-symmetric divisor $\overline{\text{orb}(\langle r \rangle)}$ as a group of orthogonal transformations.

Lemma 68:

Let $G, r, \Sigma_{\langle r \rangle}$ as above. Then the fan $\Sigma_{\langle r \rangle}$ is the fan of $(\mathbb{P}^1)^{n-1}$.

Proof:

Let $\Delta_{\langle r \rangle} = \Delta_1 \times \dots \times \Delta_k$ be the root system of $\Sigma_{\langle r \rangle}$. By induction $\Sigma_{\langle r \rangle}$ is a product of two and one-dimensional root systems. Assume that Δ_1 corresponds to a two-dimensional irreducible factor.

If there exists a non-trivial subspace of $(r)_{\mathbb{R}}^{\perp}$ invariant under $\text{Stab}_r(G)$, then it has dimension 1 or $n - 2 > 2$ (this is only possible for $G \cong A_n, D_n$).

Therefore Δ_1 cannot be stabilized by $\text{Stab}_r(G)$ and there exists a reflection $S \in \text{Stab}_r(G)$ such that

$$S(\Delta_1) \neq \Delta_1.$$

We can choose a base

$$r_1, r_2, \dots, r_{n-1} \text{ of } \Delta_{\langle r \rangle}$$

such that r_1, r_2 is a base of Δ_1 and that r_1, r_2 are orthogonal to all other elements of the base but not orthogonal to each other (otherwise Δ_1 is reducible; cf. [G-B], p. 56).

Without loss of generality we assume $S(r_1) \notin \Delta_1$.

Since $S(r_1) \in \Delta_{\langle r \rangle}$, it follows that $S(r_1)$ is orthogonal to Δ_1 . Now S is an orthogonal mapping, thus it also follows that

$$S(r_2) \notin \Delta_1$$

and therefore

$$S(\Delta_1) \perp \Delta_1.$$

But if E_S is the hyperplane stabilized pointwise by S , then

$$\dim(E_S \cap (\Delta_1)_{\mathbb{R}}) \geq 1.$$

This contradicts $S(\Delta_1) \perp \Delta_1$. $\square$

By Lemma 68 we are able to apply the same methods as in the four-dimensional classification.

In particular we can use the results obtained there for the classical groups.

Theorem 67 is now a consequence of the following

Proposition 69:

If Σ is defined by the root system Δ of an irreducible Coxeter group G of dimension $n \geq 5$ and if $\Sigma_{\langle r \rangle}$ is the fan of $(\mathbb{P}_1)^{n-1}$ (r as in Lemma 46), then Σ cannot be complex-symmetric.

Proof:

We consider the 6 different types of Coxeter groups. In each case we will consider the orthogonal projection $\tilde{\Delta}$ of Δ on $(r)_{\mathbb{R}}^{\perp}$. If $\Sigma_{\langle r \rangle}$ is the fan of $(\mathbb{P}_1)^{n-1}$, then there is a set

$$A := \{\pm v_1, \dots, \pm v_{n-1}\} \subset \tilde{\Delta}$$

such that $v_1, \dots, v_{n-1}$ are mutually orthogonal and

$$g(A) = A \quad \forall g \in \text{Stab}_r(G).$$

Case 1: $G \cong A_n$

In this case it is known that r can be chosen arbitrarily. It follows from Lemma 52 that a set A with the required properties does not exist.

Case 2: $G \cong B_n$

Let Δ be the root system of B_n specified as in 2.2.

By Lemma 54 the set A cannot contain vectors of the form

$$\pm e_i \pm e_j \in \tilde{\Delta}.$$

So the only possibility for A is

$$A = \{\pm e_1, \dots, \pm e_{n-1}\}.$$

Therefore $\langle e_1, \dots, e_{n-1} \rangle$ has to be a cone in $\Sigma_{\langle r \rangle}$.

By Lemma 56 it follows that

$$\langle e_1 + e_n, \dots, e_{n-1} + e_n, e_n \rangle$$

is a cone in Σ .

But then by Lemma 57 the fan Σ cannot be complex-symmetric.

Case 3: $G \cong D_n$

Let Δ be the root system of D_n specified as in 2.2.

By Lemma 60 the only vectors $v \in \tilde{\Delta}$ which have the property that $g(v)$ and v are either orthogonal or linearly dependent for all $g \in \text{Stab}_r(G)$ are

$$v = e_{n-1} - e_n \quad \text{and} \quad v = -e_{n-1} + e_n$$

This does not define a complete system.

Case 4: $G \cong E_8$

By Lemma 46 there exists $r \in \Delta$ (root system of E_8) such that

$$\text{Stab}_r(G) \cong E_7.$$

Now we can proceed as in the proof of Lemma 63: The group of fan automorphisms of the fan corresponding to $(\mathbb{P}_1)^7$ has order

$$2^7 \cdot 7! = 645\,120.$$

But the order of E_7 is 2 903 040 ([G-B], p. 80).

Therefore E_7 cannot act as subgroup of the fan automorphisms of $(\mathbb{P}^1)^7$. Thus E_8 does not define a complex-symmetric fan.

Case 5: $G \cong E_7$

The root system Δ of E_7 can be regarded in the hyperplane $(e_7 + e_8)^\perp_{\mathbb{R}}$ of $\mathbb{R}^8$. We may assume that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 6\} \cup \{\pm(e_7 - e_8)\} \\ \cup \{\pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^6 (-1)^{\nu(i)} e_i) \mid \sum_{i=1}^6 \nu(i) \text{ odd}\} \quad ([\text{Bou}], \text{ p. 264})$$

and that

$$\pi = \{r_1, \dots, r_7\} \\ = \{\frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + \dots + e_7), e_1 + e_2, e_2 - e_1, e_3 - e_2, \dots, e_6 - e_5\}$$

is a base of Δ ([Bou], p. 264).

Now consider the vector $r = e_7 - e_8$.

This vector is orthogonal to $r_2, \dots, r_7$, but not orthogonal to r_1 .

By Proposition 45 the group $\text{Stab}_r(G)$ is generated by

$S_{r_2}, \dots, S_{r_7}$, and we have

$$\text{Stab}_r(G) \cong \mathcal{O}_6 \quad (\text{Coxeter graph: } \begin{matrix} r_3 & r_4 & r_5 & r_6 & r_7 \\ o_3 & o_4 & o_5 & o_6 & o_7 \end{matrix}) \\ r_2^o$$

Now $\tilde{\Delta}$ consists of the projection of Δ on $(e_7 - e_8, e_7 + e_8)^\perp_{\mathbb{R}} = (e_7, e_8)^\perp_{\mathbb{R}}$. Therefore we have

$$\tilde{\Delta} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 6\} \cup \{\pm \frac{1}{2} \sum_{i=1}^6 (-1)^{\nu(i)} e_i \mid \sum_{i=1}^6 \nu(i) \text{ odd}\}.$$

The vectors of the form $e_i \pm e_j$ cannot be elements of A .

We choose $k \neq i, j$ ($1 \leq i, j, k \leq 6$). Then $S_{e_i - e_k} \in \text{Stab}_r(G)$ and $S_{e_i - e_k}(e_i \pm e_j) = e_k \pm e_j$ and $e_i \pm e_j$ are neither orthogonal nor linear dependent.

For vectors $v = \frac{1}{2} \sum_{i=1}^6 (-1)^{\nu(i)} e_i$, the condition " $\sum_{i=1}^6 \nu(i)$ odd" implies that there exists k such that $(-1)^{\nu(k)} = 1$ and that there exists j such that $(-1)^{\nu(j)} = -1$.

Since $\text{Stab}_r(G)$ acts transitively on the set $e_1, \dots, e_6$, we may assume

$$v = \frac{1}{2}(e_1 - e_2 + \sum_{i=3}^6 (-1)^{\nu(i)} e_i).$$

But

$$S_{e_1 - e_2}(v) = \frac{1}{2}(e_2 - e_1 + \sum_{i=3}^6 (-1)^{\nu(i)} e_i)$$

is not a multiple of v and not orthogonal to v . Therefore we have shown that $\tilde{\Delta}$ does not contain a set A with the required properties.

Case 6: $G \cong t_6$

We may assume that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 5\} \cup \{\pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^5 (-1)^{v(i)} e_i) \mid \sum_{i=1}^5 v(i) \text{ even}\}$$

is the root of system of E_6 in

$$(e_6 - e_7, e_7 + e_8)_{\mathbb{R}}^{\perp} \subset \mathbb{R}^8 \text{ and that}$$

$$u = \{r_1, \dots, r_6\} = \{\frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + \dots + e_7), e_1 + e_2, e_2 - e_1, \dots, e_5 - e_4\}$$

is a base for Δ ([Bou], p. 260).

The vector $r := \frac{1}{2}(e_1 + \dots + e_5 - e_6 - e_7 + e_8) \in \Delta$ is orthogonal to all base vectors except r_2 .

Thus $\text{Stab}_r(G) \cong A_5$ is generated by $S_{r_1}, S_{r_3}, \dots, S_{r_6}$.

Lemma 70:

Under the assumptions of Case 6 let $\tilde{\Delta}$ be the set of projections of elements of Δ on $(r)_{\mathbb{R}}^{\perp}$. Then we have

$$\begin{aligned} \tilde{\Delta} = S_1 \cup S_2 \cup S_3 = \{ \pm(e_i - e_j) \mid 1 \leq i < j \leq 5 \} \cup \{ \pm(e_i + e_j - \frac{1}{2}r) \} \\ \cup \{ \pm w_1, \dots, \pm w_5 \}. \end{aligned}$$

where

$$w_i := -(e_1 + \dots + e_5) + e_8 - e_7 - e_6 + 2e_i, \quad i = 1, \dots, 5.$$

Proof:

An easy calculation shows that all vectors of $\tilde{\Delta}$ are orthogonal to $r, e_6 - e_7, e_7 + e_8$.

Vectors in S_1 are elements of Δ and vectors in S_2 are obtained as the orthogonal projection of the vectors $\pm(e_i + e_j)$ on $(r)_{\mathbb{R}}^{\perp}$.

For a vector v of the form

$$v = \frac{1}{2} \left(\sum_{i=1}^5 (-1)^{v(i)} e_i \right) - e_6 - e_7 + e_8$$

the condition " $\sum_{i=1}^5 v(i) \text{ even}$ " yields three different possibilities:

Either $(-1)^{v(i)} = +1 \forall i = 1, \dots, 5$

or $(-1)^{v(i)} = +1 \Leftrightarrow i = i_1, i_2, i_3 \quad (i_j \in \{1, \dots, 5\})$

or $(-1)^{v(i)} = +1 \Leftrightarrow i = i_1 \in \{1, \dots, 5\}.$

In the first case we have $v = r$, so this case need not be considered.

In the second case we have

$$r = v + (c_k + c_\ell) \quad (1 \leq k, \ell \leq 5, k, \ell \neq i_1, i_2, i_3).$$

So the projection of these vectors is of the form $\pm(c_i + c_j - \frac{1}{2}r)$.

In the third case we have vectors of the form w_{i_1} .

This completes the proof of the Lemma. $\square$

Now for $k \neq i, j$ ($1 \leq k \leq 5$) we have that

$$S_{e_i - e_k}(\pm(e_i - e_j)) = \pm(e_k - e_j)$$

is not a multiple of $\pm(e_i - e_j)$ nor orthogonal to $\pm(e_i - e_j)$.

Therefore vectors of the form $\pm(e_i - e_j)$ cannot be elements of A .

Since $S_{e_i - e_k}(r) = r$, we can use $S_{e_i - e_k}$ for vectors of the form $\pm(e_i + e_j - \frac{1}{2}r)$ as well.

Finally we have

$$S_{e_i - e_k}(w_i) = w_k,$$

and w_i and w_k are not orthogonal.

Thus a set A with the required properties does not exist in the case $G \cong E_6$.

This completes the proof of Proposition 69.

Thus Theorem 67 is proved by induction.

As an immediate Corollary of Theorem 67 we obtain

Corollary 71:

Let X be a smooth minimal complete complex-symmetric torus embedding of dimension n . Then

$$X \cong (\mathbb{P}_1)^n. \quad \square$$

Corollary 72:

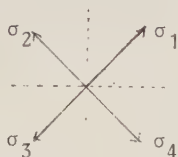
Let X be a complete complex-symmetric torus embedding of dimension n .

Then X is projective algebraic (T -equivariantly) and can equivariantly be blown down to $(\mathbb{P}_1)^n$. $\square$

We will finish this paper by mentioning the singular case. At first we give an example of a singular torus embedding in dimension two.

Consider the complete fan Σ with

$$\text{Sk}(\Sigma) = \{(1,1), (-1,1), (-1,-1), (1,-1)\}.$$



Since $\det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2 > 1$, this is a complete torus embedding X_Σ with four singular points belonging to the maximal-dimensional cones. We show that this example defines a complex-symmetric torus embedding for $T \cong (\mathbb{C}^*)^2$.

At first it is clear by Lemma 21 that symmetries for points in the open T -orbit exist since the fan is centrally symmetric by construction.

Moreover, since X_Σ is smooth except for the fixed points of T , the condition of Theorem 24 is sufficient for the codimension-one orbits of T in X_Σ , i.e. symmetries for points in $\text{orb}(\sigma_i)$ exist, if and only if there exists a fan automorphism φ such that

$$\varphi(\sigma_i) = \sigma_i \quad \text{and} \\ \text{one eigenvalue of } \varphi \text{ is } -1.$$

It is easy to see that

$$\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{resp. } \varphi = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix})$$

are fan isomorphisms with the required properties for $i = 1, 3$ (resp. for $i = 2, 4$).

In order to find symmetries for the fixed points of the T -action on X_Σ note that these symmetries have to be induced by translations since the corresponding fan isomorphism is the identity (cf. Remark following Proposition 5). However, in contrast to the case of non-singular torus embeddings it is not clear that there are symmetries for the fixed points of the T -action.

Consider σ_1 . Then the dual cone $\mathcal{V}_1$ is generated by the vectors $(-1,1)$ and $(1,1)$ (i.e. σ_1 is self-dual). The semigroup $\mathcal{V}_1 \cap M$ is generated by $(1,1), (-1,1)$ and $(0,1)$. Thus the affine variety corresponding to $\mathcal{V}_1$ can be identified with

$$X = \{(x,y,z) \in \mathbb{A}^3 \mid xy = z^2\}.$$

The T -action on X is defined by

$$(\lambda, \mu)(x, y, z) = (\lambda \mu x, \lambda^{-1} \mu y, \mu z).$$

Now define $\lambda_0 = 1$, $\mu_0 = -1$. We then have

$$(\lambda_0, \mu_0)(x, y, z) = (-x, -y, -z).$$

This defines a symmetry for the fixed point $(0,0,0)$ of the T -action on X .

Since X is an open affine subvariety of the torus embedding X_Σ and the action of T on X can be extended to an action of T on X_Σ by definition, we have found a symmetry for one fixed point in X_Σ . Symmetries for the other fixed points can be obtained similarly.

Thus the torus embedding X_Σ is complex-symmetric.

Final Remark:

By the Remark following Lemma 31 the skeleton corresponding to the fan of a singular complex-symmetric torus embedding also defines the root system of a Coxeter group.

Now the Coxeter group corresponding to the Example X_Σ above is $A_1 \times A_1$.

Since X_Σ is not a product of one-dimensional torus embeddings, we see that in the singular case reducible Coxeter groups perhaps do not correspond to reducible torus embeddings.

Moreover, there are at least two non-isomorphic varieties which have $A_1 \times A_1$ as corresponding Coxeter group, while it follows from our classification that there is at most one smooth variety corresponding to a fixed Coxeter group.

So in the singular case a classification seems to be much more difficult.

However, it seems possible to classify two- and three-dimensional torus embeddings with methods similar to those used in 1.4 and 2.1. The crucial point is whether a T -stable divisor in a singular complex-symmetric torus embedding is again complex-symmetric. If this can be proved, inductive arguments can be applied to consider the general case.

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